Propositional Self-Reference and Modalities

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Abstract

This paper considers modal self-referential sentences and argues that they generate semantic paradoxes similar to the Liar. The sources of related antinomies are similar as in the case of the Liar-sentence, namely self-referentiality and the **T**-scheme, additionally supplemented by some principles connecting modalities and truth. In the Appendix at the end of the paper, the dual logic is employed for constructing the Truth-Teller Paradox and its modal counterparts.

1 A basic modal system

Self-referential propositions (sentences, statements; I will use these words as equivalents of ‘proposition’) say something about themselves, that is, if *A* is a sentence, it is self-referential, provided that it attributes a given property *P* to itself. For instance, the sentence ‘the sentence in single quotes in the second sentence in this paper is false’ is self-referential – in fact, it codes the famous Liar-sentence. Similarly, the statement ‘the sentence in single quotes in this sentence occurring in this paper is self-referential’ – it is the Truth-Teller sentence. Some propositional self-referentialities, lead to paradoxes, for example, the Liar-sentences, other, for instance ‘This sentence consists of six words’ are paradox-free. I will consider propositions of the type

(*) This sentence is (not) **M**,

where **M** expresses a modality. For example, the statements ‘This sentence is not necessary’, ‘This proposition is impossible’, ‘This proposition is possible’, ‘This propositions is necessary’ etc. fall under (*).

I assume the system (**D1**) of modal logical principles consisting of four points α, β, γ, δ and relations between them (this system constitutes the modal logical system as presented in Woleński 2008). Interpreting α as $\blacksquare A$, β as $\blacksquare\neg A$, γ as $\blacklozenge A$ and δ as $\blacklozenge\neg A$, we have the following facts

(1) $\neg(\alpha \wedge \beta)$ (α and β are contrary);
(2) $(\alpha \Rightarrow \gamma)$ (α entails γ; γ is subordinated to α);
(3) $(\beta \Rightarrow \delta)$ (β entails δ; δ is subordinated to β);
(4) $(\alpha \Leftrightarrow \neg\delta)$ (α and δ are contradictory);
(5) $(\beta \Leftrightarrow \neg\gamma)$ (β and γ are contradictory);

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(6) $(\gamma \vee \delta)$ (γ and δ are complementary);
(7) $(\blacksquare A \Leftrightarrow \neg \blacklozenge \neg A)$ ($\blacksquare A$ is definable as $\neg \blacklozenge \neg A$);
(8) $(\blacklozenge A \Leftrightarrow \neg \blacksquare \neg A)$ ($\blacklozenge A$ is definable as $\neg \blacksquare \neg A$);
(9) $(\beta \Leftrightarrow \blacksquare \neg A)$ (β is definable as $\blacksquare \neg A$);

Due to (7)–(9) negation standing before the symbols $\blacksquare$ or $\blacklozenge$ is always eliminable. We can consider (1)–(9) as validities of **(D1)**-logic – in a sense, **(D1)** constitutes the minimal modal logic. This formal basis of further considerations is supplemented by classical propositional logic (**PC – D1** is a modal extension of **PC**) and three additional theses

(10) $\forall A(A \Leftrightarrow \mathbf{T}A)$, where **T** is the truth predicate 'is true'; the naive **T**-scheme);
(11) $\forall A(\blacksquare A \Rightarrow \mathbf{T}A)$ (or $\neg(\blacksquare A \wedge \neg \mathbf{T}A)$);
(12) $\forall A \neg(\blacksquare \neg A \wedge \mathbf{T}A)$.

Roughly speaking, (11) means that no necessity is not true, but (12) that no impossibility is true – assuming that 'not true' means 'false', we obtain a more impressive reading, namely that no necessity is false, and that every impossibility is false). I do not enter into the problem of what it means that a sentence is necessary – identification of necessities with logical tautologies is a possible interpretation of the box $\blacksquare$.

The simplest illustration of **(D1)**-logic is provided by alethic modalities. Interpret $\blacksquare$ as necessary that, $\blacksquare\neg$ as impossible (necessary not) that, $\blacklozenge$ as possible that, and $\blacklozenge\neg$ as possible not (I will use *de dicto* versions, for example $\blacksquare A$, and *de re* versions, for example 'A is necessary' as equivalents). We have that necessity and impossibility are contrary, necessity entails possibility, impossibility entails possibility not, necessity and possibility not are contradictory, impossibility and possibility are contradictory, possibility and possibility not are complementary, necessity is equivalent to non-possibility not, possibility is equivalent to non-necessity not, and impossibility is equivalent to necessity not (also necessity is equivalent to impossibility not).

2 Modal self-referential paradoxes

Consider the sentence

(13) The sentence signified by (13) in this paper is impossible.

Replace (13) by c (the same letter will be employed in the further examples of this kind (I hope that this stipulation does not lead to any confusion). Simple transformations (the conventions that (13) = c, and if c = B, then $c \Leftrightarrow B$), give

(14) $c \Leftrightarrow \blacksquare \neg c$

By (10) we obtain

(15) $\mathbf{T}c \Leftrightarrow \blacksquare\neg c$.

However, (15) contradicts (11). Consider now

(16) The sentence signified by (16) in this paper is not necessary.

Replace (16) by c. Simple transformations give

(17) $\mathbf{T}c \Leftrightarrow \neg\blacksquare c$;

(18) $\blacksquare c \Leftrightarrow \neg\mathbf{T}c$.

However, (18) contradicts (11).

Clearly, (13) recalls the Liar Paradox (the Liar, for brevity) obtainable from the sentence 'This sentence is false', but (16) – the antinomy generated by the sentence 'This sentence is not true' (the equivalence of 'is false' and 'is not true' can be rejected). However, these analogies, that is, between impossibility and falsehood as well as between non-necessity and being not true, are somehow simplified, because relations between modalities and truth (falsity) are more complicated. In order to have a more refined picture, consider the extended system of modalities (**D2**). It arises from **(D1)** by adding four new points, namely ν, κ, λ, μ. We interpret ν as $\alpha \vee \beta$ ($\blacksquare A \vee \blacksquare\neg A$), κ as $\mathbf{T}A$, λ as $\neg\mathbf{T}A$ ($\mathbf{F}A$ – A is false, if $\neg\mathbf{T}A \Leftrightarrow \mathbf{F}A$, that is the principle of bivalence in the form $\forall A(\mathbf{T}A \vee \mathbf{F}A)$ is valid in our logic; in fact, it is the case because we assume the classical system), μ as $\blacklozenge A \wedge \neg\blacksquare A$ ($\blacklozenge A \wedge \blacklozenge\neg A$ or $\neg\blacksquare\neg A \wedge \neg\blacksquare A$). The formula at the point ν can be read 'A is modally determined, that is either A is necessary or A is impossible', but the formula at the point μ – 'A is modally not determined, that is neither necessary nor impossible'. This last reading can be questioned, because one might argue that every point in (**D2**) can be considered as expressing a modal determination. Hence, another, and perhaps a more intuitive reading, suggests that ν should be read as 'A is strongly modally determined' and μ as 'A is weakly modally determined', but I will use the former reading. According to the tradition, δ is understood as 'A is contingent (non-necessary)', but if so, one should remark that μ – appears also as a candidate to be read as expressing contingency (double possibility). Other possible readings of μ are given by 'A is accidental' or 'A is modally indifferent'. To complete the terminological remarks, note that truth and falsity belong to the modal variety according to the formal theory generated by (**D2**).

(**D2**) leads to new modal principles. In particular, we have (I do not list all cases)

(19) $(\alpha \Rightarrow \kappa)$ ($\blacksquare A \Rightarrow \mathbf{T}A$; necessity implies truth);
(20) $(\beta \Rightarrow \lambda)$ ($\blacksquare\neg A \Rightarrow \mathbf{F}A$; impossibility implies falsity);
(21) $(\kappa \Rightarrow \gamma)$ ($\mathbf{T}A \Rightarrow \blacklozenge A$; truth implies possibility);
(22) $(\lambda \Rightarrow \mu)$ ($\mathbf{F}A \Rightarrow \neg\blacksquare A$; falsity implies non-necessity);
(23) $\neg(\kappa \wedge \lambda)$ ($\neg(\mathbf{T}A \wedge \mathbf{F}A)$; truth and falsity are contradictory);
(24) $\neg(\nu \wedge \mu)$ ($\neg((\blacksquare A \vee \blacksquare\neg A) \wedge (\blacklozenge A \wedge \blacklozenge\neg A))$; modal determinacy and modal indeterminacy are contradictory);
(25) $\neg(\gamma \Rightarrow \alpha)$ ($\neg(\mathbf{T}A \Rightarrow \blacksquare A)$; truth does not entail necessity);

(26) $\neg(\lambda \Rightarrow \beta)$ $(\neg(\mathbf{F}A \Rightarrow \blacksquare\neg A)$; falsity does not entail impossibility);
(27) $\neg(\gamma \Rightarrow \kappa)$ $(\neg(\blacklozenge A \Rightarrow \mathbf{T}A)$; possibility does not entail truth);
(28) $\neg(\delta \Rightarrow \lambda)$ $(\neg(\neg\blacksquare A \Rightarrow \mathbf{F}A)$; non-necessity does not entail falsity);
(29) $\neg\vdash^{\mathrm{D2}}\nu$ $(\neg\vdash^{\mathrm{D2}}(\blacksquare A \vee \blacksquare\neg A)$; modal determinacy is not provable in (**D2**);
(30) $\neg\vdash^{\mathrm{D2}}\mu$ $(\neg\vdash^{\mathrm{D2}}(\blacklozenge A \wedge \blacklozenge\neg A)$; modal indeterminacy is not provable in (**D2**);
(31) $(\nu\vee\mu)$ $((\blacksquare A\vee\blacksquare\neg A)\vee(\blacklozenge A\wedge\blacklozenge\neg A)$; A is modally determined or non-determined).

Returning to paradoxes, the Liar ('this sentence is false') is related to λ. The paradox displayed by (13) can be called the Impossibility-Teller, but the Non-Necessity-Teller is the puzzle expressed by (16). Thus, we have the sequence $\langle\beta,\lambda,\delta\rangle$ such that a self-referential instantiation of each member of it generate paradoxes. On the other hand, the sequence $\langle\alpha,\kappa,\gamma\rangle$ reminds us of the Truth-Teller problem, that is the question whether the Truth-Teller sentence as self-referential leads to a paradox or not; α – the Necessity-Teller, γ – the Possibility Teller, and this means that we can expect that the related self-referential sentences are paradox-free (I return to this question in the Appendix). What about sentences "This sentence is not modally determined" (The Modal Non-Determinacy-Teller) and "This sentence is modally determined" (The Modal Determinacy-Teller). Intuitively, if a sentence is the disjunction of a non-paradoxical statement and paradoxical one, it should not be paradoxical – this concerns ν. On the other hand, if a sentence is a conjunction of a paradox and a non-paradox, like μ, it should be regarded as paradoxical.

In fact, it can be proved that the sentence

(32) The sentence signified by (32) in this paper is not modally determined.

is paradoxical. Omitting some simple steps occurring on the occasion of (13) and used earlier, we obtain

(33) $\mathbf{T}c \Leftrightarrow \blacklozenge c \wedge \blacklozenge\neg c$,

which entails

(34) $\mathbf{T}c \Rightarrow \blacklozenge\neg c$.

By contraposition, we get (I employ the equivalence $\neg\mathbf{T}A \Leftrightarrow \mathbf{F}A$)

(35) $\neg\blacklozenge\neg c \Rightarrow \mathbf{F}c$,

equivalent by definition based on (**D1**) to

(36) $\blacksquare c \Rightarrow \mathbf{F}c$.

However, the last formula contradicts (11), which asserts that no necessity can be false. Thus, (32) in its formal representation, that is, the formula $\blacklozenge c \wedge \blacklozenge\neg c$ under the proviso of the equivalence (33) is paradoxical, because the second conjunct $(\blacklozenge\neg c)$ leads to a contradiction modulo $\mathbf{T}c \Leftrightarrow \blacklozenge c \wedge \blacklozenge\neg c$. Of course, the same result is derivable, if $\blacklozenge\neg c$ is to be replaced by the formula $\neg\blacksquare c$. Summing up, we have two mutually symmetric varieties, the first $\langle\alpha,\kappa,\gamma,\nu\rangle$ and the second – $\langle\beta,\lambda,\delta,\mu\rangle$. The former contains Modal-Tellers free of paradoxes, but the latter includes Modal-Tellers generating self-referential paradoxes – both divide (**D2**) into two symmetric parts.

3 Tarski's diagnosis of the Liar and modal self-referential paradoxes

Tarski (see Tarski 1944, p. 672), following Leśniewski, diagnosed the Liar in the following way (see Pleitz 2018 for an extensive analysis of the Liar and its problems):

(I) We have implicitly assumed that the language in which the antinomy is constructed contains, in addition to its expressions, also the names of these expressions, as well as semantic terms such as the term "true" referring to the sentences of this language; we have also assumed that all sentences which determine the adequate usage of this term can be asserted in the language. A language with these properties will be called "*semantically closed.*"

(II) We have assumed that in this language the ordinary laws of logic hold.

(III) We have assumed that we can formulate and assert in our language an empirical premise such as the statement (13) [in the present paper] which has occurred in our argument

The first premise can be split into two parts: (a) the language, say **L,** contains expressions as well as their names (the semantic closeness), and (b) the **T**-scheme, regulating the use of the predicate 'is true', holds universally. Consequently, semantic terms can be used self-referentially, that is, that the predicates 'is true' and 'is false' can refer to sentences in which they occur. To sum up, the Liar is generated by the following assumptions (I skip the point (III) occurring in the quotation from Tarski):

(A) Sentential self-referentiality;

(B) the **T**-scheme as universal, that is (10);

(C) Classical (ordinary) logic.

Consequently, we have three different options for resolving the Liar. Firstly, we can eliminate self-referentiality; secondly, reject or modify the **T**-scheme as a basic constraint for using the predicate 'is true', and, thirdly, change logic from classical to another system. Tarski chose the first alternative, that is, he excluded self-referentiality. This recipe immediately leads to modifying the **T**-scheme. Its new version, is recorded by the formula

(37) $\mathbf{T}\ulcorner A\urcorner \Leftrightarrow A^*$, for any sentence A^* of a language **L**,

where the expression $\ulcorner A\urcorner$ names the sentence A in the metalanguage **ML**, but the expression A^* refers to the translates of this sentence into **ML**.

The **T**-scheme is essential for the Liar in two respects. Firstly, it occurs in the derivation of the paradox in question, explicitly in Tarski's version, and implicitly in other reasoning known from the antiquity. Secondly, the Liar is considered as critical for the classical (semantic) definition of truth. Hence, if someone likes to save this definition, he or she must solve the Liar. Thus, Tarski's solution of the paradox simultaneously excluded self-reference, preserved the distinction of **L** and **ML** and modified the naive **T**-scheme (that is, (10)) by accepting (37). Clearly, if we exclude sentential modal self-reference the above stated paradoxes of modality disappear. On the other hand, the **T**-scheme does not regulate the use of modal words – their, so to speak, minimal meaning stems from (**D2**). In fact, in order to derive modal self-referential paradoxes I used (except (10)) additional rules, namely (12) and (13). The **T**-scheme connects modalities and truth in derivations of particular modal paradoxes. In other words, if (10) (I neglect (37)) functions somehow internally in the case of the Liar, its role in modal paradoxes is rather external. If we place $\mathbf{T}A$ in the point α, $\mathbf{T}\neg A$ in the point β, A in the point κ, and $\neg A$ in the point λ, $\neg\mathbf{T}\neg A$ in the point γ, $\neg\mathbf{T}A$ in the point δ, $\mathbf{T}A \vee \mathbf{T}\neg A$ in the point ν, and $\neg\mathbf{T}\neg A \wedge \neg\mathbf{T}A$ in the point μ we obtain a new interpretation of (**D2**), generating some logical laws concerning the truth-predicate **T**. Thus, truth is interpreted as a kind modality – in particular, **T**, under this interpretation, falls under $\blacksquare$, although it does not mean that every truth is necessary. What about the **T**-scheme in (**D2**)-logic of truth? The implication

(38) $\mathbf{T}A \Rightarrow A$,

is justified by the principle analogical to (19), but its converse, that is, the formula

(39) $A \Rightarrow \mathbf{T}A$,

does not hold in this logic. Consequently, the **T**-scheme must be added as a new principle, which cuts the truth-logic to the points α, β, κ, λ, ν. Accordingly, we have $\mathbf{T}\neg A \Leftrightarrow \neg\mathbf{T}A \Leftrightarrow \mathbf{F}A$, that is, not-truth and falsity are equivalent (I employed this fact earlier. If we reject (39) the Liar disappears (see Turner 1991, p. 24, Halbach 2011, Chapters 13 and 15). In a general perspective, the **T**-scheme does not constrain the concept of truth in Tarski's way, but we still need to keep the language/metalanguage distinction in order to block various semantic paradoxes, for instance, the heterologicality puzzle.

The formula (39), and thereby, the entire **T**-scheme also plays a crucial role in the modal paradoxes. I will consider (13). Now we cannot replace A by $\mathbf{T}c$. We have only

(40) $\mathbf{T}c \Rightarrow c$;
(41) $c \Leftrightarrow \blacksquare\neg c$ (that is, (13)).

These two formulas give

(42) $\mathbf{T}c \Rightarrow \blacksquare\neg c$;
(43) $\neg \blacksquare \neg c \Rightarrow \neg\mathbf{T}c$;

(44) $\blacklozenge c \Rightarrow \neg \mathbf{T}c$,

which is not contradictory, because if A is possible, it can still be false as well. Consequently, (38) does not produce a paradox, if it acts together with (13). However, if we add (39), we get (10) and obtain the Impossibility-Teller Paradox. The Non-Necessity-Teller and the Modal-Indeterminacy Teller can be treated in a similar manner. Their solution also consists in making a distinction between the language and the metalanguage or resigning from the **T**-scheme.

4 Other kinds of modalities

Modalities are not exhausted by the alethic case. We have also deontic modalities (it is obliged, it is permitted, etc.), doxastic and epistemic modalities (it is asserted, it is supposed, it is known, etc.) or erotetic modalities (it is asked, it is questioned). All satisfy (**D1**), but the principles (19) and (20) are rather exceptional and this means that (**D2**) cannot be considered as the general logic of modalities. In particular, deontic modalities violate the rule (19), because 'it is obliged that A' does not entail 'it is true that A'. I see no way to derive self-referential modal-deontic referential paradoxes analogous to the alethic ones, except perhaps in the epistemic case. Interpret now $\blacksquare A$ as "it is known that A". According to the classical definition of knowledge as true justified belief we have

(45) $\blacksquare A \Rightarrow \mathbf{T}A$.

Consider now

(46) The sentence signified by (46) is not known, that is
(47) $c \Leftrightarrow \neg \blacksquare c$.

Using the earlier strategy, we have

(48) $\mathbf{T}c \Leftrightarrow \neg \blacksquare c$;
(49) $\mathbf{T}c \Rightarrow \neg \blacksquare c$;
(50) $\neg \blacksquare c \Rightarrow \mathbf{T}c$

Combining (45) with contraposition of (49) entails

(51) $\blacksquare c \Rightarrow \mathbf{T}c \wedge \neg \mathbf{T}c$,

which shows that (46) leads to a paradox. The paradoxes for $\blacksquare\neg c$ and $\blacklozenge c \wedge \blacklozenge\neg c$ can be proved similarly. Note, however, that (45) is not obvious, because this rule is motivated not by the (**D2**)-logic, but the classical definition of knowledge. It is also contested due to the Gettier problem, but I will not enter into details (see Shope 1983). We can also prove the paradox of omniscience, which admits the converse of (45), because the omniscient being knows the entire truth (every true proposition). Hence, for every A, if A is true, than it is knowable (formally, $\forall A(\mathbf{T}A \Rightarrow \blacksquare A)$).

5 Final remarks

The Liar is commonly considered as a semantic paradox, that is, closely related to semantic concepts, in this case, the notion of truth. What about self-referential modal paradoxes? Are they semantic or not? Clearly, necessity, possibility, knowledge, etc. are not purely semantic categories. In this sense, related antinomies are not quite similar to the Liar. In particular, whereas the latter is derivable via the **T-**scheme, but without additional premises (excluding findings about the letter A and corresponding numberings), modal antinomies require this scheme plus some additional assumptions, like (11), (12) or (45). A very important circumstance consists in the fact that all additional premises link modal concepts with truth. If a connection of modality with truth is absent, no self-referential paradox arises. Take the sentence 'it is justified that'. It implies neither $\mathbf{T}A$ nor its negation $\neg\mathbf{T}A$. Similarly, obligation and other deontic modalities are separated from truth (it is Hume's generalized thesis, see Woleński 2006 for a general treatment of this issue – Hume argued that ought is not derivable from is). The diagnosis that the **T**-scheme and modality/truth connections are important for the paradoxes considered, is confirmed by other Liar-like antinomies, for instance, that of analyticity and meaningfulness (see Woleński 2016, Woleński 2018). Finally, let me make four remarks. Firstly, Kant had a good intuition dividing propositions according to their modality into assertoric, problematic (possibilities) and apodictic (necessities), at least in this respect that he considered the first category as a kind of modalities. Secondly, the concept of truth is understood widely in this paper in such a way that tautologies are included in the set of truths. I do not decide whether every necessary truth is tautological or not, but (see above) I admit that such identity can be assumed. Yet we can formulate paradoxes for 'is (not) tautological (not)' in a similar manner as for necessity. Thirdly, since we have no general theory of modality, concerning all of its kinds, all assertions about the scope of the paradoxical terrain must be taken as empirical hypotheses. Finally, let me add that, because the Liar (see above) concerns the concept of truth, its consequences are much more dramatic than inconsistencies in the case of self-referential modal statements. As I already noted, the **T**-scheme does not directly concern the meaning of modal words, the paradoxes considered above are rather conceptual puzzles than real difficulties. On the other hand, the recipe is the same as in the case of the Liar, namely the exclusion of self-reference and preserving the language/metalanguage distinction. On the other hand, we do not need to transfer modalities into **ML.** Fourthly, similarly as in the case of the Liar, one can formulated strengthened versions of modal sentential paradoxes, generated by the sentence 'this sentence is not **M** or syntactically incorrect'.

6 Appendix. Dual logic and paradoxes

Using classical logic (more precisely, a modal extension of classical propositional calculus as given by the (**D2**)-logic) was one of the crucial assumptions in my previous considerations. However, this presumption requires an addition. Speaking about classical logic we have in mind the system satisfying the bivalence principle ($\forall A(\mathbf{T}A \vee \mathbf{F}A)$). Yet we tacitly assume that this logic has truth as the designated value, that is, A is a tautology if and only if it is true for all valuations of propositional variables occurring in it (true in all models, true in all possible worlds, universally valid, etc.). But what about logic with falsity considered as the designated value? Such systems were anticipated by Jan Łukasiewicz, and constructed by Jerzy Słupecki and his students (see Słupecki, Bryll, Wybraniec-Skardowska 1971; the rejection logic), and Ryszard Wójcicki (see Wójcicki 1973, the dual logic). The main idea is that A is a tautology if and only if A is false for all valuations of its variables. Consequently counter-tautologies of the standard logical system (let us use this label) become tautologies in the new logic. Yet this logic is perfectly two-valued. Moreover, B logically follows from A in the new logic if and only if B cannot be false, provided that A is true.

In the next I will use (I follow Wójcicki 1973 and Woleński 1995) the dual logic (*d*logic). We define the duality of formulas by the following settings

(52)(a) $d(p) = p$, for any propositional variable;
(b) $d(A) = \neg A$;
(c) $d(A \wedge B) = d(A) \vee d(B)$;
(d) $d(A \vee B) = d(A) \wedge d(B)$;
(e) $d(A \Rightarrow B) = \neg d(A) \vee d(B)$;
(f) $d(A \Leftrightarrow B) = d(A \Rightarrow B \wedge B \Rightarrow A)$.

The original motivation for the dual logic (in the case of the rejection logic is the same) consisted in formalizing the concept of rejection, without using in the object language the phrase 'is rejected'. Intuitively, and in the standard metatheory, if A is rejected, then $A \wedge B$ is rejected, and if A is rejected and B is also rejected, then $A \vee B$ is rejected as well. In *d*logic, these ideas are formally represented by (52c) and (52e). Let symbol *dCn* refer to the dual consequence operation. We have

(53) $A \in dCnX \Leftrightarrow dA \in CndX$,

which shows the connection between *dCn* and *Cn* – this dependency allows to translate inferences using *Cn* into ones employing *dCn*. In fact, *dCn* satisfies the same general axioms as *Cn*, for instance, it is monotonic, finitary and idempotent.

At first, I will formulate the Truth-Teller Paradox. The reasoning requires the *d***T**-scheme given by the formula

(54) $\mathbf{F}A \Leftrightarrow {}^{\mathrm{d}}A$.

Intuitively, the formula (54) says that it is rejected that A is false if and only if A. Intuitively (in our standard metalanguage), we reject A provided that A is false. Since the formula $A \Leftrightarrow {}^{d}B$ is defined as $\neg A \wedge B \vee A \wedge B$, (54) becomes (I simplify reasoning by simultaneously combining the standard logic and *d*logic)

(55) $(\mathbf{T}A \wedge A) \vee (\mathbf{F}A \wedge \neg A)$;

Consider now the sentence

(56) The sentence signified by (56) in this paper is true.

Formally, we obtain via using *d*logic, conventions adapted from previous settings, (54), and after transformation of (55),

(57) $c \vee \neg c$.

However, this step finishes the reasoning, because tautologies are inconsistent in *d*logic (another construction of this paradox is to be found in Mortensen, Priest 1981; the authors assume a logic with truth-value gaps). Now consider

(58) The sentence signified by (58) in this paper is necessary,

That is, formally

(59) $\mathbf{T}c \Leftrightarrow {}^{d}c$.

After some simple steps we obtain

(60) $(\mathbf{T}c \wedge \blacksquare c) \vee (\mathbf{F}c \wedge \neg \blacksquare c)$;
(61) $(\mathbf{T}c \vee \mathbf{F}c) \wedge (c \vee \neg c)$.

Since (61) is a tautology, as the conjunction of two tautologies, the demonstration of the Necessity-Teller is completed. Proofs that the Possibility-Teller and the Modal Determinacy-Teller lead to paradoxes are analogical.

The reading of (54) and (55) requires some further comments. Recall that the formula $\mathbf{F}A \Leftrightarrow {}^{d}A$ means that the equivalence $\mathbf{F}A \Leftrightarrow A$ is rejected. This is not directly indicated in (55). However, using duality of *Cn* and *dCn*, we can translate formulas of the *d*logic into formulas of the standard logic. Thus, (55) becomes (**REJ** means 'is rejected').

(62) $(\mathbf{T}A \wedge A) \vee (\mathbf{F}A \wedge \neg A) \in \mathbf{REJ}$,

that is

(63) $(\mathbf{T}A \vee \mathbf{F}A) \wedge (A \vee \neg A) \in \mathbf{REJ}$.

Applying to the expression denoted by the letter *c* (related to (56) and (58)), we obtain

(64) $(\mathbf{T}c \vee \mathbf{F}c) \wedge (c \vee \neg c) \in \mathbf{REJ}$,

which means that tautology is rejected, that is, contradiction asserted. This finishes the proof that (56) and (58) are paradoxical.

As it could be expected, the role of the *d***T**-scheme is crucial in derivations of the Tellers-Paradoxes. Due to mixing both kinds of logic, we do not need analogues of (11) and (12). I do not claim that the *d*logic is particularly intuitive. All cases of rejection in the object language ((54) is an example) can be translated in the standard metalanguage with rejection – saying that 'if $A \Rightarrow B$ is rejected and $\neg A$ is rejected, B is rejected as well' or 'if $A \vee B$ is rejected, A is rejected and B is rejected'. One can say "Well, since logic as such does prefer neither standard nor *d*logic, why should we use the former and understand the connectives according to the former, even if we reject the statements?" The answer is that our language and the consequence operation are facts-oriented. This attitude leads to recognizing assertions as more fundamental than rejections. Hence, the Liar was discovered very early, but the Truth-Teller not very far ago. As far as the issue concerns self-referential modal paradoxes, they play no particular role in our understanding of modalities. On the other hand, they confirm that self-referential use of words expressing semantic properties can lead to difficulties. Unfortunately, there exists no general theory, which would delineate "safe" self-referential properties of expressions from "dangerous" ones, in order to generalize the last conclusion of the main text, even if we assume that second attributes are of a semantic character.

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