Quantified Temporal Alethic Boulesic Deontic Logic

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Abstract

The purpose of this paper is to develop a set of quantified temporal alethic boulesic deontic systems. Every system in this class consists of five parts: a 'quantified' part, a temporal part, a modal part (an alethic part), a boulesic part and a deontic part. Separately, all these parts, except the boulesic part, have been studied extensively, but there are no systems in the literature that combine them all. So, all systems in this paper are new. The 'quantified part' consists of relational predicate logic with identity, where the quantifiers are, in effect, a kind of possibilist quantifiers that vary over every object in the domain. The alethic part includes two types of modal operators, for absolute and historical necessity and possibility. By 'boulesic logic', I mean the logic of the will; it treats 'willing' ('consenting', 'rejecting', 'indifference' and 'non-indifference') as a kind of modal operator. Deontic logic is the logic of norms; it deals with such concepts as ought, permitted and forbidden. I will investigate some possible relationships between these different parts, and consider various principles that include more than one type of logical expression. Every system is described both semantically and proof theoretically. I use a kind of $T \times W$ semantics to describe the systems semantically, and semantic tableaux to describe them proof theoretically. I prove that every tableau system in the paper is sound and complete with respect to its semantics. Finally, I consider some examples of valid and invalid sentences and arguments, show how one can use semantic tableaux to prove their validity or invalidity, and try to illustrate the philosophical usefulness of the systems developed in the paper.

Keywords: Quantified modal logic, Modal logic, Temporal logic, Deontic logic, Boulesic logic, Semantic tableaux.

1 Introduction

In this paper, I will introduce a set of quantified temporal alethic boulesic deontic systems. Every system in this class includes five parts: a 'quantified' part, a temporal part, a modal part (an alethic part), a boulesic part and a deontic part. Separately, each of these parts, except the boulesic part, has been investigated thoroughly. Some interactions between them have also been explored. Connections between alethic and deontic logic, between temporal, alethic and deontic logic, and between predicate and modal logic have, for example, been investigated (see below for references). However, as far as I know, there are no systems in the literature that combine them all. Hence, all systems in this paper are new. Since the different parts, except the boulesic part, are well-known, I will focus on the combination of the various components. Every system will be characterised semantically using a kind of $T \times W$ semantics. According to this approach, both worlds and times are basic and truth is relativised to world-moment pairs. Consequently, a sentence may be true at some world-moment pairs and false at others. The $T \times W$ approach is mentioned by [123], [152] and [157], among others. I will develop a set of semantic tableau systems and prove that they are sound and complete with respect to their semantics.

The alethic part of our systems includes two types of modal operators for absolute and historical necessity and possibility, respectively: $\mathbb{U}$ (absolute necessity), $\mathbb{M}$ (absolute possibility), $\Box$ (historical necessity) and $\Diamond$ (historical possibility). For introductions to (alethic) modal logic, see, for example, [25], [26], [44], [54], [59], [61], [64], [84], [85], [94], [96], [129], [147], [148], [149] and [164].

Every system includes several temporal operators, for example, $\mathbb{A}$ (always), $\mathbb{S}$ (sometimes), $\mathbb{G}$ (always in the future), $\mathbb{F}$ (sometime in the future), $\mathbb{H}$ (always in the past) and $\mathbb{P}$ (sometime in the past). For more on temporal logic, see, for example, [20], [40], [55], [68], [95], [119], [122] and [112].

Deontic logic is the logic of norms. Every system in this paper includes deontic operators such as **O** (ought) and **P** (permitted) that can be used to symbolise various normative propositions. For some introductions to deontic logic, see, for example, [5], [7], [62], [67], [71], [76], [77], [105] and [153].

By 'boulesic logic' I mean a new kind of logic, the logic of the will; it treats 'willing' ('consenting', 'rejecting', 'indifference' and 'non-indifference') as a kind of modal operator. Every system includes five boulesic operators $\mathcal{W}$, $\mathcal{A}$, $\mathcal{R}$, $\mathcal{I}$ and $\mathcal{N}$. $\mathcal{W}$, $\mathcal{A}$, $\mathcal{R}$, $\mathcal{I}$ and $\mathcal{N}$ are sentential operators that take individual terms and formulas as arguments and give formulas as values. The sentence $\mathcal{W}_d B$ reads 'individual d wants it to be the case that B', the sentence $\mathcal{A}_d B$ reads 'd accepts that (it is the case that) B', or 'd consents to the state of affairs (the idea) that B', the sentence $\mathcal{R}_d B$ reads 'd rejects (disapproves, objects to, condemns) (the state of affairs that) B', the sentence $\mathcal{I}_d B$ reads 'd is indifferent towards (the state of affairs that) B', and the sentence $\mathcal{N}_d B$ reads 'd is non-indifferent towards (the state of affairs that) B'. Even though boulesic logic is new, there have been some vaguely similar attempts to develop a kind of 'intentional' logic, see, for example, [28], [29], [46], [100], [102] and [131]; see also [66], Chapter 10, [92] and [101]. The approach in this paper is, however, quite different. According to this approach, almost nothing of interest follows from the proposition that someone wants something (or has some other boulesic attitude towards something), unless we assume that this individual is (perfectly) rational or wise. However, *if* we assume that some individual is (perfectly) rational, we can derive all sorts of interesting consequences from the claim that this individual wants something (or has some other boulesic attitude towards something). Exactly what follows will depend on the interpretation of the concept of rationality and on what conditions we choose to impose on the so-called boulesic accessibility relation in our semantic models (see Section 3.3). For more on non-temporal boulesic logic, see [124]. For more on the concept of rationality, see, for example, [107] and [31].

The 'quantified part' of the systems consists of relational predicate logic with identity. The quantifiers are, in effect, a kind of 'possibilist' quantifiers that vary over every object in the domain and the domain is the same in every world-moment pair. Every system includes a universal quantifier, Π ('everything'), and a particular quantifier, Σ ('something'). In every system, we can also define a pair of 'actualist' quantifiers in terms of the possibilist quantifiers and an existence predicate. Intuitively, the actualist quantifiers vary over everything that exists in a world-moment pair, that is, that exists in a world at a particular moment in time. For some views on how to combine modal logic and predicate logic, see, for example, [17], [41], [47], [59], [63], [64], [78], [84], [85], [97], [98], [114], [118], [137], [142] and [141].

I will investigate some possible relationships between these different parts of our systems, and consider various principles that include more than one type of logical expression. Some interactions of this kind have been explored before. Logicians have, for example, developed systems that combine alethic and deontic logic, temporal, alethic and deontic logic, and predicate and modal logic. Some of the first attempts to combine deontic logic and alethic modal logic can be found in several essays by Anderson (see [1], [2], [3], [4]). Another early contribution is [58]; see also [87].

Many philosophers and logicians have developed logical systems that include temporal, alethic and deontic elements, see, for example, [43], [11], [12], [13], [14], [150], [143], [144], [10] and [9]. Chellas ([43]) also adds a modal logic of action to his systems. For more ideas on how to combine deontic logic with temporal logic, see, for example, [15], [27], [33], [34], [35], [36], [37], [38], [42], [44], Chapter 6, [52], [53], [146] and [8]. See also [18], [19], [32], [70] and [83].

Systems that combine modal and temporal logic with a kind of action logic have been developed by researchers within the stit-paradigm. Sometimes these systems are combined with deontic logic. For more on stit-logic, see, for example, [16], [22], [73], [79], [80], [81], [110], [113], [131], [158], [159], [160], [161] and [162]. For more on how to combine modal and temporal logic, see, for example, [45], [50], [145] and [165]. See [108] for an early attempt to combine various systems.

There are many good reasons, both technical and philosophical, to be interested in the results in this paper. Since all systems are new, there are good logical reasons to be interested in them. I use semantic tableau in this paper. Most logicians who have tried to combine different branches of logic, such as, for example, temporal logic and deontic logic, have used axiomatic techniques. Tableau systems are often more user-friendly. It is often easier to prove theorems, establish the validity or invalidity of various principles and arguments, and find countermodels in tableau systems. Our symbolic apparatus might also be useful in linguistics and computer science.

I cannot discuss all the philosophical reasons to be interested in our systems in detail, but let me mention five points to illustrate the usefulness of our technical results.

First, we appear to need systems of this kind to prove that certain statements that are intuitively valid are valid. Consider the following example:

E1. It is (absolutely) necessary that if a perfectly rational individual *x* wants it to be the case that *A* sometime in the future and it is (historically) necessary that it is always going to be the case that if *A* then *B*, then *x* wants it to be the case that *B* sometime in the future.

This sentence is intuitively valid. If it will be the case that *A* sometime in the future and it is historically necessary that it is always going to be the case that if *A* then *B*, then it is inevitable that it will sometime in the future be the case that *B*. Hence, if someone is perfectly rational and she wants it to be the case that *A* sometime in the future, she also wants it to be the case that *B* sometime in the future given that it is historically necessary that it is always going to be the case that if *A* then *B*. In Section 7, I will show that we can prove this sentence in every system that includes the tableau rule $T - \Box\mathcal{W}$. However, we cannot prove this proposition in any other system in the literature (at least not without assuming some implicit premises).[1]

Second, we can use the systems to find countermodels to some propositions that are intuitively invalid. Consider the following sentence:

E2. If an individual *x* wants it to be the case that *x* sometime in the future will be a citizen of Great Britain and it is (historically) necessary that it is always going to be the case that if *x* is a citizen of Great Britain then *x* is a citizen of Europe, then *x* wants it to be the case that *x* sometime in the future will be a citizen of Europe.

Even though **E1** is intuitively valid, **E2** is intuitively invalid. If someone is *not* perfectly rational, she may want *A* even though she does not want every necessary condition for *A*. In Section 7, I will show how we can prove that **E2** is invalid in the class of all models and how one can use semantic tableaux to construct countermodels to invalid sentences.

Third, we appear to need systems of the kind in this paper to prove that certain arguments that are intuitively valid indeed are valid.

Consider the following example:

E3. *P*1. It is (absolutely) necessary that if no perfectly rational individual accepts that you will rape someone in the future, then it ought to be (the case) that it is always going to be the case that you do not rape someone.

*P*2. Everyone who is perfectly rational wants it to be the case that it is never going to be the case that you rape someone.

C. Hence, it is not permitted that you will rape someone in the future.

This argument is intuitively valid. It appears to be necessary that the conclusion is true if the premises are true. However, we cannot establish this in any systems in the literature. In Section 7, I will show how we can use a semantic tableau to prove that this argument is valid in the class of all models. To be able to prove this, we need to use all parts of our systems. Note that the argument includes an alethic expression ('absolutely necessary'), quantifier expressions ('no' and 'everyone'), boulesic expressions ('accepts' and 'wants'), temporal expressions ('will in the future' and 'it is always go-

[1] Note that when we say that some individual *c* wants (or accepts or . . .) *A*, we usually mean that *c* wants (accepts, etc.) *A* in an all-things-considered sense in this paper. For example, *c* might not feel like going to the dentist; nevertheless, all-things-considered she wants to go. Going to the dentist is a means to an end, namely, healthy teeth. Accordingly, when we say that *c* wants (or accepts, etc.) *A*, we do not necessarily mean that *c* wants (or accepts, etc.) *A* 'in itself'; in fact, *c* might dislike *A*, even though she wants *A* to be the case because *A* is a necessary means to or conditions for something else that she wants 'in itself'. In other words, it is possible for *c* to want (accept, etc.) *A* in an all-things-considered sense even though *c* does not like every aspect of *A* or every consequence of *A* and even if *c* has some desire (a prima facie desire) for not-*A*. For more on this, see [124].

ing to be the case that') and deontic expressions ('permitted' and 'ought'). Hence, to prove that the argument is valid, we need a system that includes all of these parts.

Fourth, our systems can be used to analyse and prove some interesting principles in ethics and metaethics, for example, the principle of internalism. There are many kinds of internalism, but according to one version of this principle the following proposition is true:

(I). It is absolutely necessary that if someone x is perfectly rational, then if it ought to be the case that A then x wants it to be the case that A.

This principle can be symbolised in the following way in our systems: $\mathbb{U}\Pi x(Rx \rightarrow (\mathbf{O}A \rightarrow W_x A))$. This formula can be proved in every system that includes the tableau rule $T - \mathbf{O}\mathcal{W}$. Consequently, it is valid in the class of all $C - \mathbf{O}\mathcal{W}$-models (by the soundness results in Section 6). For more on internalism, see, for example, [23], [24] and [151]. See also Section 5.

Fifth, our systems can be used in the development of whole ethical systems. Our logical systems seem to be particularly well suited to developing a kind of Kantian ethics, but they might also be interesting to, for example, various ideal observer theorists (see Section 5 for more on this), constructivists, moral idealists, contractualists and divine will theorists.

Let me briefly illustrate how the systems in this paper can be used to analyse several Kantian theses, for example, the so-called ought implies can principle, the hypothetical imperative, and the idea that for a perfectly rational individual 'I ought' and 'I will' are equivalent. In some systems we can even prove that these principles are valid. These examples clearly illustrate that our systems can be used in the development of a kind of Kantian ethics.

It is generally agreed that Kant thought that ought implies can. He expresses this idea in several places in his works, see, for example, [89] 6:47 ('duty commands nothing but what we can do') and [89] 6:62 ('We *ought* to conform to it, and therefore we must also *be able* to'). So, someone ought to do something only if she can do it according to Kant; we do not have any obligations that are impossible to fulfil. According to one interpretation of this principle, this means (or at least entails) that it is absolutely necessary that ought implies (historical) possibility. Hence, it is absolutely necessary that if it ought to be the case that A then it is (historically) possible that A. This principle can be symbolised in the following way in our systems: $\mathbb{U}(\mathbf{O}A \rightarrow \Diamond A)$, and this schema can be proved in every system that includes the tableau rule $T - \mathbf{O}\Diamond$. Consequently, it is valid in the class of all $C - \mathbf{O}\Diamond$-models (by the soundness results in Section 6).[2]

Kant introduced the concept of a hypothetical imperative. In *Grundlegung zur Metaphysik der Sitten*, he defines this concept in the following way:

[2] Since Kant many other philosophers and logicians have accepted the ought implies can principle. Kant is probably the most famous defender of this thesis, but he was not the first to accept it, see, for example, [120], Book I, Chapter V, VIII: 'Impossibilities are incapable of Obligation; ... no Man can be conceiv'd to have enjoin'd impossible Duties in a Law...'. For more on the ought-can principle, see, for example, [49], [57], [82], [91], [99], [104], [109], [111], [133], [134], [138], [139], [140], [154] and [163].

> 'Who wills the end, wills (so far as reason has decisive influence on his actions) also the means which are indispensably necessary and in his power' and "'If I fully will the effect, I also will the action required for it" is analytic'. ([88], p. 45; English translation in [115], pp. 80–81.)

Since Kant, there has been debate about what is the logical form of a hypothetical imperative. I will not enter into this debate in the present paper; I just want to point out that it is possible to symbolise several different interpretations of the concept in our systems and mention one of the most plausible. According to this interpretation, it is universally necessary that, for every x, if x is perfectly rational, then if x wants it to be the case that A and it is necessary that if A then B, then x wants it to be the case that B. This reading can be symbolised in the following way: $\mathbb{U}\Pi x(Rx \rightarrow ((\mathcal{W}_x A \wedge \Box(A \rightarrow B)) \rightarrow \mathcal{W}_x B))$. This formula can be proved in every system that includes $T - \Box\mathcal{W}$. Since a system that includes $T - \Box\mathcal{W}$ is sound with respect to the class of all $C - \Box\mathcal{W}$-models, the sentence is valid in this class of models (by the soundness results in Section 6). I discuss this principle in more detail in [124]. Universal necessity implies historical necessity. Hence, we can also prove the following version of the hypothetical imperative: $\mathbb{U}\Pi x(Rx \rightarrow ((\mathcal{W}_x A \wedge \mathbb{U}(A \rightarrow B)) \rightarrow \mathcal{W}_x B))$. In fact, this sentence can be proved in every system in this paper and hence it is valid in the class of all models.[3]

According to [116], p. 223, "'I ought" is equivalent to "I will" for a rational agent as such' for a Kantian (see also [115], p. 26). This idea can be symbolised in our systems in the following way: $\mathbb{U}\Pi x(Rx \rightarrow (\mathbf{O}A \leftrightarrow \mathcal{W}_x A))$, which can be read as 'It is absolutely necessary that for every x, if x is perfectly rational then it ought to be the case that A if and only if (iff) x wants it to be the case that A'.[4] This schema is a theorem in every system that includes the tableau rules $T - \mathbf{O}\mathcal{W}$ and $T - \mathcal{W}\mathbf{O}$, and it is valid in the class of models that satisfy $C - \mathbf{O}\mathcal{W}$ and $C - \mathcal{W}\mathbf{O}$. It is not obvious that Kant himself would accept this principle—he seems to think that there are no 'oughts' for a perfectly rational individual ([115], p. 78)—but the difference between these positions is not great. For if 'I ought' is equivalent to 'I will' for a rational agent, then every 'I ought' can, in principle, be 'eliminated'. In any case, the principle is clearly Kantian in spirit.

Consequently, we can symbolise and prove at least three versions of three important Kantian ethical principles. Of course, these principles are not the only ones that would be included in a more fully developed Kantian ethical system and much more could be said about them, but the discussion above is sufficient for our purposes in this paper and clearly shows the usefulness of our systems.

I conclude that we have very good reasons to be interested in the systems in this paper.

The paper consists of seven main sections. Section 2 deals with the syntax and Section 3 with the semantics of our systems. In Section 4, I describe the proof theory of our logics, while Section 5 includes some examples of theorems. Section 6 contains

[3] For more information about the hypothetical imperative, see, for example, [21], [30], [39], [51], [52], Chapter 5, [60], [65], [69], [72], [74], [75], [93], [103], [132], [125], [126], [127], [128], [155] and [156].

[4] Note that this formula entails **(I)** above, that is, it entails a kind of internalism.

soundness and completeness proofs for every system. Finally, in Section 7, I consider some examples of valid and invalid sentences and arguments.

2 Syntax

2.1 Alphabet

Terms: (i) A set of variables x_1, x_2, x_3 …. (ii) A set of constants (rigid designators) k_{d_1}, k_{d_2}, k_{d_3}, …. **Predicates:** (iii) For every natural number $n > 0$, n-place predicate symbols P^1_n, P^2_n, P^3_n …. (iv) The monadic existence predicate E, and the monadic rationality predicate R. (v) The dyadic identity predicate (necessary identity) $=$. **Connectives:** (vi) The primitive truth-functional connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (material implication) and $\leftrightarrow$ (material equivalence). **Operators:** (vii) The alethic operators $\mathbb{U}$, $\mathbb{M}$, $\Box$ and $\Diamond$. (viii) The temporal operators $\mathbb{A}$, $\mathbb{S}$, $\mathbb{G}$, $\mathbb{F}$, $\mathbb{H}$ and $\mathbb{P}$. (ix) The deontic operators **O** and **P**. (x) The boulesic operators $\mathcal{W}$, $\mathcal{A}$, $\mathcal{R}$, $\mathcal{I}$ and $\mathcal{N}$. **Quantifiers:** (xi) The (possibilist) quantifiers Π and Σ. **Parentheses:** (xii) The brackets) and (.

x, y and z … stand for arbitrary variables, a, b, c … for arbitrary constants, and s and t for arbitrary terms (with or without primes or subscripts). For more on the set of constants, see Section 3.1. I will use F_n, G_n, H_n … for arbitrary n-place predicates and I will omit the subscript if it can be read off from the context.

2.2 Language

The language $\mathcal{L}$ is defined in the following way: (i) Any constant or variable is a term. (ii) If $t_1, \ldots, t_n$ are any terms and P is any n-place predicate, $Pt_1 \ldots t_n$ is an atomic formula. (iii) If t is a term, Et ('t exists') is an atomic formula and Rt ('t is perfectly rational') is an atomic formula. (iv) If s and t are terms, then $s = t$ ('s is identical with t') is an atomic formula. (v) If A and B are formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A \leftrightarrow B)$. (vi) If A is a formula, so are $\mathbb{U}A$ ('it is universally [or absolutely] necessary that A'), $\mathbb{M}A$ ('it is universally [or absolutely] possible that A'), $\Box A$ ('it is [historically] necessary that A') and $\Diamond A$ ('it is [historically] possible that A'). (vii) If B is a formula, so are $\mathbb{A}B$ (it is always the case that B), $\mathbb{S}B$ (it is sometimes the case that B), $\mathbb{G}B$ (it is always going to be the case that B), $\mathbb{F}B$ (it will some time [in the future] be the case that B), $\mathbb{H}B$ (it has always been the case that B) and $\mathbb{P}B$ (it was some time [in the past] the case that B). (viii) If B is any formula and t is any term, then $\mathcal{W}_t B$ ('t wants it to be the case that (desires that) B'), $\mathcal{A}_t B$ ('t accepts that (consents to the idea that, approves that, tolerates that, is willing that) (it is the case that) B'), $\mathcal{R}_t B$ ('t rejects (disapproves, objects to, condemns) (the state of affairs that) B'), $\mathcal{I}_t B$ ('t is indifferent towards (the state of affairs that) B') and $\mathcal{N}_t B$ ('t is non-indifferent towards (the state of affairs that) B') are formulas. (ix) If A is any formula and x is any variable, then $\Pi x A$ ('for every [possible] x: A') and $\Sigma x A$ ('for some [possible] x: A') are formulas. (x) If A is a formula, then **O**A ('it ought to be the case that A') and **P**A ('it is permitted that A') are formulas. (xi) Nothing else is a formula.

The concepts of bound and free variables, and open and closed formulas, are defined in the usual way. A, B, C … stand for arbitrary formulas, and Γ, Φ … for finite sets of closed formulas. $A[a_1, \ldots, a_n/x_1, \ldots, x_n]$ is the result of replacing every free occurrence of x_1 by a_1, and …, and every free occurrence of x_n by a_n in A. $A[a_1, \ldots, a_n/x_1, \ldots, x_n]$ will be abbreviated as $A[a_1, \ldots, a_n/\vec{x}]$. $A[t/x]$ is the formula obtained by substituting t for every free occurrence of x in A. The definitions are standard. Brackets around formulas are usually dropped if the result is not ambiguous.

Definition 1 *The definitions below should be treated as pure metalogical abbreviations. However, in parentheses I will indicate how the new symbols might be interpreted informally.* ***Deontic operators:*** $\mathbf{F}A$ *('it is forbidden that A')* $=_{df} \neg\mathbf{P}A$*;* $\mathbf{K}A$ *('it is optional that A')* $=_{df} (\mathbf{P}A \wedge \mathbf{P}\neg A)$*;* $\mathbf{N}A$ *('it is non-optional that A')* $=_{df} \neg\mathbf{K}A$*.* ***Temporal operators:*** $\underline{\mathbb{G}}A$ *('it is and it is always going to be the case that A')* $=_{df} (A \wedge \mathbb{G}A)$*;* $\underline{\mathbb{H}}A$ *('it is and it has always been the case that A')* $=_{df} (A \wedge \mathbb{H}A)$*;* $\underline{\mathbb{F}}A$ *('it is or it will some time in the future be the case that A')* $=_{df} (A \vee \mathbb{F}A)$*;* $\underline{\mathbb{P}}A$ *('it is or it has some time in the past been the case that A')* $=_{df} (A \vee \mathbb{P}A)$*.* ***Actualist quantifiers:*** $\forall xA$ *('for every existing x A')* $=_{df} \Pi x(Ex \to A)$ *and* $\exists xA$ *('for some existing x A')* $=_{df} \Sigma x(Ex \wedge A)$*.*

3 Semantics

3.1 Models

Definition 2 ***(Models)*** *A model* $\mathcal{M}$ *is a relational structure* $\langle D, W, T, <, \mathfrak{R}, \mathfrak{A}, \mathfrak{S}, v\rangle$*, where D is a non-empty set of individuals (the domain), W is a non-empty set of possible worlds, T is a non-empty set of times, < is a binary relation on T (< is a subset of* $T \times T$*),* $\mathfrak{R}$ *is a ternary alethic accessibility relation (*$\mathfrak{R}$ *is a subset of* $W \times W \times T$*),* $\mathfrak{A}$ *is a four-place boulesic accessibility relation (*$\mathfrak{A}$ *is a subset of* $D \times W \times W \times T$*),* $\mathfrak{S}$ *is a ternary deontic accessibility relation (*$\mathfrak{S}$ *is a subset of* $W \times W \times T$*), and v is an interpretation function.*

$\mathfrak{R}$ is used in the definition of the truth conditions for sentences that begin with the alethic operators $\Box$ and $\Diamond$, $\mathfrak{S}$ is used in the definition of the truth conditions for sentences that begin with the deontic operators **O** and **P**, $\mathfrak{A}$ is used in the definition of the truth conditions for sentences that begin with the boulesic operators $\mathcal{W}$, $\mathcal{A}$, $\mathcal{R}$, $\mathcal{I}$ and $\mathcal{N}$, and $<$ is used to define the truth conditions for sentences that begin with the temporal operators. Intuitively, $\tau < \tau'$ says that the time τ is before the time τ', $\mathfrak{R}\omega\omega'\tau$ says that the possible world ω' is alethically (historically) accessible from the possible world ω at the time τ, $\mathfrak{S}\omega\omega'\tau$ says that the possible world ω' is deontically accessible from the possible world ω at the time τ, and $\mathfrak{A}\delta\omega\omega'\tau$ says that the possible world ω' is boulesically accessible (acceptable) to the individual δ in (or relative to) the possible world ω at the time τ, or that δ accepts ω' in (or relative to) ω at τ.[5]

[5]In this paper, we treat $\mathfrak{A}$ as primitive. However, it might in principle be possible to define this relation. Here are some possible definitions: ω' is acceptable to δ in ω at τ iff the utility of ω' for δ at τ is positive, or above a certain threshold or as high as possible, or iff δ does not prefer any other possible world to ω' in ω at τ, or …. The important thing for our purposes in this paper is that all definitions of this kind are compatible with the semantics we use. The models are also consistent with the proposition that different

The valuation function v assigns every constant c an element $v(c)$ of D, and each world-moment pair, $\langle\omega,\tau\rangle$, and n-place predicate, P, a subset, $v_{\omega\tau}(P)$ (the extension of P in ω at τ), of D^n. In other words, $v_{\omega\tau}(P)$ is the set of n-tuples that satisfy P in the world ω at time τ (in the world-moment pair $\langle\omega,\tau\rangle$). Hence, every constant is a kind of rigid designator: it refers to the same individual in every world-moment pair. Nonetheless, the extension of a predicate may change from world-moment pair to world-moment pair and it may be empty in a world-moment pair. Let $\mathcal{M}$ be a model. Then the language of $\mathcal{M}$, $\mathcal{L}(\mathcal{M})$, is obtained by adding a constant k_d, such that $v(k_d) = d$, to the language for every member $d \in D$. Hence, every object in the domain of a model has at least one name in our language, but several different constants may refer to one and the same object.

The predicate R has a special interpretation in our systems. 'Rc' says that c is *perfectly rational, perfectly reasonable* or *perfectly wise*. If $v(c)$ is in the extension of R in the possible world ω at the time τ, this means that $v(c)$ is perfectly rational, reasonable or wise in ω at τ. Exactly what this means will depend on the conditions we impose on the boulesic accessibility relation $\mathfrak{A}$ (Section 3.3). R functions as an ordinary predicate. Hence, an individual δ may be in R's extension in one world-moment pair even though δ is not in R's extension in every world-moment pair. Accordingly, the fact that an individual δ is perfectly rational, reasonable or wise in *one* world-moment pair does not *entail* that δ is perfectly rational, reasonable or wise in *every* world-moment pair. In Section 3.3, we will see what happens if we add the extra assumption that every perfectly rational individual is necessarily perfectly rational (the semantic condition $C-UR$ guarantees that this is the case: see Table 16). In the light of the definitions of the truth conditions for the boulesic sentences (see Section 3.2, 18–22), it should be obvious that R plays an important role in our systems. It will become even clearer when we introduce the various tableau rules in Section 4.

Let A be a closed boulesic formula of the form $\mathcal{W}_tB$, $\mathcal{A}_tB$, $\mathcal{R}_tB$, $\mathcal{I}_tB$ or $\mathcal{N}_tB$. Then, the matrix of A is constructed in the following way. Let m be the least number greater than every n such that x_n occurs bound in B. From left to right, replace every occurrence of an individual constant with x_m, x_{m+1}, etc. The result is the formula's matrix. Here are some examples: the matrix of $\mathcal{W}_aPc$ is $\mathcal{W}_{x_1}Px_2$; the matrix of $\mathcal{A}_cPaa$ is $\mathcal{A}_{x_1}Px_2x_3$; the matrix of $\mathcal{W}_c(Fa \to Gbc)$ is $\mathcal{W}_{x_1}(Fx_2 \to Gx_3x_4)$; the matrix of $\mathcal{A}_a\Pi x_1(Fx_1 \vee Gc)$ is $\mathcal{A}_{x_2}\Pi x_1(Fx_1 \vee Gx_3)$; the matrix of $\mathcal{W}_c\mathcal{W}_d\Sigma x_2Px_2$ is $\mathcal{W}_{x_3}\mathcal{W}_{x_4}\Sigma x_2Px_2$, etc. The valuation function assigns extensions to matrices of this kind. If M is any matrix of the form $\mathcal{W}_tB$, $\mathcal{A}_tB$, $\mathcal{R}_tB$, $\mathcal{I}_tB$ or $\mathcal{N}_tB$ with free variables $x_1,\ldots,x_n$, then $v_{\omega\tau}(M) \subseteq D^n$. Note that M always includes at least one free variable. Let M be a matrix where x_m is the first free variable in M and a_m is the constant in $M[a_1,\ldots,a_n/\vec{x}]$ that replaces x_m. Then the truth conditions for closed boulesic formulas of the form $M[a_1,\ldots,a_n/\vec{x}]$, when $v_{\omega\tau}(Ra_m) = 0$, are defined in terms of the extension of M in ω at τ (see condition 2 in Section 3.2 below).[6]

$v_{\omega\tau}(=) = \{\langle d,d\rangle : d \in D\}$ (the extension of the identity predicate is the same in every possible world at every moment in time (in a model)). It follows that all identities

individuals might have different reasons for accepting ω' in ω at τ. It is an interesting question whether or not it is possible to define $\mathfrak{A}$, but for our purposes in this paper, we do not have to answer this question.

[6]See [117], Chapter 1–2, for more on matrices.

(and non-identities) are both absolutely and historically necessary, as well as eternal. The existence predicate E functions as an ordinary predicate. The extension of this predicate may vary from one world-moment pair to another. 'Ec' is true in a possible world at a time iff $v(c)$ exists in this world at this time.

3.2 Truth conditions and some semantic concepts

Let $\mathcal{M}$ be any model $\langle D, W, T, <, \mathfrak{R}, \mathfrak{A}, \mathfrak{S}, v\rangle$. Let $\omega \in W$, $\tau \in T$ and let A be a well-formed sentence in $\mathcal{L}$. Then $\mathcal{M}, \omega, \tau \Vdash A$ is an abbreviation of 'A is true in ω at τ in $\mathcal{M}$' (or 'A is true in the pair $\langle\omega, \tau\rangle$ in $\mathcal{M}$'). $\mathcal{M}, \omega, \tau \nVdash A$ just in case it is not true that $\mathcal{M}, \omega, \tau \Vdash A$. Note that $\mathcal{M}, \omega, \tau \nVdash A$ iff $\mathcal{M}, \omega, \tau \Vdash \neg A$. '$\forall\omega' \in W$' is read as 'for all possible worlds ω' in W'; '$\exists\omega' \in W$' is read as 'for some possible world ω' in W', etc. The truth conditions for various sentences in $\mathcal{L}$ can now be defined in the following way (the truth conditions for the omitted sentences are straightforward):

1. $\mathcal{M}, \omega, \tau \Vdash Pa_1 \dots a_n$ iff $\langle v(a_1), \dots, v(a_n)\rangle \in v_{\omega\tau}(P)$.
2. Let M be a matrix where x_m is the first free variable in M and a_m is the constant in $M[a_1, \dots, a_n/\vec{x}]$ that replaces x_m. Then the truth conditions for closed boulesic formulas of the form $M[a_1, \dots, a_n/\vec{x}]$, when $v(a_m)$ is not an element in $v_{\omega\tau}(R)$, are as follows: $\mathcal{M}, \omega, \tau \Vdash M[a_1, \dots, a_n/\vec{x}]$ iff $\langle v(a_1), \dots, v(a_n)\rangle \in v_{\omega\tau}(M)$.
3. $\mathcal{M}, \omega, \tau \Vdash A \wedge B$ iff $\mathcal{M}, \omega, \tau \Vdash A$ and $\mathcal{M}, \omega, \tau \Vdash B$.
4. $\mathcal{M}, \omega, \tau \Vdash \mathbb{U}A$ iff $\forall\omega' \in W$ and $\forall\tau' \in T$: $\mathcal{M}, \omega', \tau' \Vdash A$.
5. $\mathcal{M}, \omega, \tau \Vdash \mathbb{M}A$ iff $\exists\omega' \in W$ and $\exists\tau' \in T$: $\mathcal{M}, \omega', \tau' \Vdash A$.
6. $\mathcal{M}, \omega, \tau \Vdash \Box A$ iff $\forall\omega' \in W$ s.t. $\mathfrak{R}\omega\omega'\tau$: $\mathcal{M}, \omega', \tau \Vdash A$.
7. $\mathcal{M}, \omega, \tau \Vdash \Diamond A$ iff $\exists\omega' \in W$ s.t. $\mathfrak{R}\omega\omega'\tau$: $\mathcal{M}, \omega', \tau \Vdash A$.
8. $\mathcal{M}, \omega, \tau \Vdash \mathbb{A}B$ iff $\forall\tau' \in T$: $\mathcal{M}, \omega, \tau' \Vdash B$.
9. $\mathcal{M}, \omega, \tau \Vdash \mathbb{S}B$ iff $\exists\tau' \in T$: $\mathcal{M}, \omega, \tau' \Vdash B$.
10. $\mathcal{M}, \omega, \tau \Vdash \mathbb{G}B$ iff $\forall\tau' \in T$ s.t. $\tau < \tau'$: $\mathcal{M}, \omega, \tau' \Vdash B$.
11. $\mathcal{M}, \omega, \tau \Vdash \mathbb{F}B$ iff $\exists\tau' \in T$ s.t. $\tau < \tau'$: $\mathcal{M}, \omega, \tau' \Vdash B$.
12. $\mathcal{M}, \omega, \tau \Vdash \mathbb{H}B$ iff $\forall\tau' \in T$ s.t. $\tau' < \tau$: $\mathcal{M}, \omega, \tau' \Vdash B$.
13. $\mathcal{M}, \omega, \tau \Vdash \mathbb{P}B$ iff $\exists\tau' \in T$ s.t. $\tau' < \tau$: $\mathcal{M}, \omega, \tau' \Vdash B$.
14. $\mathcal{M}, \omega, \tau \Vdash \mathbf{O}A$ iff $\forall\omega' \in W$ s.t. $\mathfrak{S}\omega\omega'\tau$: $\mathcal{M}, \omega', \tau \Vdash A$.
15. $\mathcal{M}, \omega, \tau \Vdash \mathbf{P}A$ iff $\exists\omega' \in W$ s.t. $\mathfrak{S}\omega\omega'\tau$: $\mathcal{M}, \omega', \tau \Vdash A$.
16. $\mathcal{M}, \omega, \tau \Vdash \Pi xA$ iff for all $k_d \in \mathcal{L}(\mathcal{M}), \mathcal{M}, \omega, \tau \Vdash A[k_d/x]$.
17. $\mathcal{M}, \omega, \tau \Vdash \Sigma xA$ iff for some $k_d \in \mathcal{L}(\mathcal{M}), \mathcal{M}, \omega, \tau \Vdash A[k_d/x]$.

18. $\mathcal{M}, \omega, \tau \Vdash \mathcal{W}_a D$ iff for all ω' such that $\mathfrak{A}v(a)\omega\omega'\tau$: $\mathcal{M}, \omega', \tau \Vdash D$, given that $v(a)$ is an element in $v_{\omega\tau}(R)$, if $v(a)$ is not an element in $v_{\omega\tau}(R)$, then $\mathcal{W}_a D$ is assigned a truth value in ω at τ in a way that does not depend on the value of D (see condition 2 above).

19. $\mathcal{M}, \omega, \tau \Vdash \mathcal{A}_a D$ iff for at least one ω' such that $\mathfrak{A}v(a)\omega\omega'\tau$: $\mathcal{M}, \omega', \tau \Vdash D$, given that $v(a)$ is an element in $v_{\omega\tau}(R)$, if $v(a)$ is not an element in $v_{\omega\tau}(R)$, then $\mathcal{A}_a D$ is assigned a truth value in ω at τ in a way that does not depend on the value of D (see condition 2 above).

20. $\mathcal{M}, \omega, \tau \Vdash \mathcal{R}_a D$ iff for all ω' such that $\mathfrak{A}v(a)\omega\omega'\tau$: $\mathcal{M}, \omega', \tau \Vdash \neg D$, given that $v(a)$ is an element in $v_{\omega\tau}(R)$, if $v(a)$ is not an element in $v_{\omega\tau}(R)$, then $\mathcal{R}_a D$ is assigned a truth value in ω at τ in a way that does not depend on the value of D (see condition 2 above).

21. $\mathcal{M}, \omega, \tau \Vdash \mathcal{I}_a D$ iff for at least one ω' such that $\mathfrak{A}v(a)\omega\omega'\tau$: $\mathcal{M}, \omega', \tau \Vdash D$ and for at least one ω' such that $\mathfrak{A}v(a)\omega\omega'\tau$: $\mathcal{M}, \omega', \tau \Vdash \neg D$, given that $v(a)$ is an element in $v_{\omega\tau}(R)$, if $v(a)$ is not an element in $v_{\omega\tau}(R)$, then $\mathcal{I}_a D$ is assigned a truth value in ω at τ in a way that does not depend on the value of D (see condition 2 above).

22. $\mathcal{M}, \omega, \tau \Vdash \mathcal{N}_a D$ iff for all ω' such that $\mathfrak{A}v(a)\omega\omega'\tau$: $\mathcal{M}, \omega', \tau \Vdash D$ or for all ω' such that $\mathfrak{A}v(a)\omega\omega'\tau$: $\mathcal{M}, \omega', \tau \Vdash \neg D$, given that $v(a)$ is an element in $v_{\omega\tau}(R)$, if $v(a)$ is not an element in $v_{\omega\tau}(R)$, then $\mathcal{N}_a D$ is assigned a truth value in ω at τ in a way that does not depend on the value of D (see condition 2 above).[7]

Π and Σ are substitutional, 'possibilist' quantifiers since the domain is the same in every possible world and every object in the domain has a name (see Section 3.1). Hence, in effect, they vary over every object in the domain.

Intuitively speaking, conditions 18–22 are interpreted in the following way. If $v(a)$ is not perfectly rational in a world-moment pair, $\mathcal{W}_a D$, $\mathcal{A}_a D$, $\mathcal{R}_a D$, $\mathcal{I}_a D$ and $\mathcal{N}_a D$ behave as ordinary predicates in this world at this time; and if $v(a)$ is perfectly rational in a world-moment pair, $\mathcal{W}_a$, $\mathcal{A}_a$ and $\mathcal{R}_a$ behave as ordinary modal operators in this world at this time. If $v(a)$ is perfectly rational in a world-moment pair, then $\mathcal{I}_a D$ is equivalent with $\mathcal{A}_a D \wedge \mathcal{A}_a \neg D$ and $\mathcal{N}_a D$ is equivalent with $\mathcal{W}_a D \vee \mathcal{W}_a \neg D$ in this world-moment pair.

Let us now define some important semantic concepts.

Definition 3 ***(Semantic concepts) Satisfiability in a model:*** *A set of sentences Γ is satisfiable in a model $\mathcal{M}$ iff there is a possible world ω and point in time τ in $\mathcal{M}$ such that every sentence in Γ is true in ω at τ.* ***Validity in a class of models:*** *A sentence A is valid in a class of models **M** iff A is true in every world at every moment of time*

[7]Note that we have to introduce all boulesic operators as primitive. If we were to restrict our systems to perfectly rational individuals, then it would be possible to use one boulesic operator as primitive (say $\mathcal{W}$) and define the other operators in terms of this operator. $\Pi x(Rx \to (\mathcal{R}_x B \leftrightarrow \mathcal{W}_x \neg B))$ is, for example, a theorem in every system in this paper. But if some individual c is not perfectly rational she may reject B even though it is not the case that she wants it to be the case that not-B. Therefore, $\mathcal{R}$ cannot be defined in terms of $\mathcal{W}$. Similar remarks apply to the other operators.

Table 1: Conditions on the alethic accessibility relation $\mathfrak{R}$

Condition	Formalisation of condition
$C - aT$	$\forall\tau\forall\omega\mathfrak{R}\omega\omega\tau$
$C - aD$	$\forall\tau\forall\omega\exists\omega'\mathfrak{R}\omega\omega'\tau$
$C - aB$	$\forall\tau\forall\omega\forall\omega'(\mathfrak{R}\omega\omega'\tau \rightarrow \mathfrak{R}\omega'\omega\tau)$
$C - a4$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{R}\omega\omega'\tau \wedge \mathfrak{R}\omega'\omega''\tau) \rightarrow \mathfrak{R}\omega\omega''\tau)$
$C - a5$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{R}\omega\omega'\tau \wedge \mathfrak{R}\omega\omega''\tau) \rightarrow \mathfrak{R}\omega'\omega''\tau)$

in every model in ***M****.* ***Logical consequence in a class of models:*** *A sentence B is a logical consequence of a set of sentences* Γ *in a class of models* ***M*** *(*$\boldsymbol{M}, \Gamma \Vdash B$*) iff for every model* $\mathcal{M}$ *in* ***M*** *and world-moment pair* $\langle\omega,\tau\rangle$ *in* $\mathcal{M}$*, if all elements of* Γ *are true in* $\langle\omega,\tau\rangle$ *(in* ω *at* τ*) in* $\mathcal{M}$*, then B is true in* $\langle\omega,\tau\rangle$ *(in* ω *at* τ*) in* $\mathcal{M}$*. If* $\boldsymbol{M}, \Gamma \Vdash B$*, then* Γ *entails B in* ***M*** *and the argument from* Γ *to B is valid in* ***M****. An argument is invalid (in* ***M****) iff it is not valid (in* ***M****).*

3.3 Conditions on models and systems of classes of models

In this section, I will consider some conditions that might be imposed on our models. These conditions concern the formal properties of the accessibility relations, the relationships between the various accessibility relations and the relationships between the accessibility relations and the valuation function. Since our models include four accessibility relations, there are 16 possible types of interactions between these relations (if we include the models where there are no interactions at all). I will consider examples of all these types.

The clauses in this section can be combined in many different ways, generating many different systems. Exactly which conditions we *should* accept seems to depend on several factors. One important factor is the interpretation of the concept of *perfect rationality* (*wisdom*). It might be the case that different conditions are plausible for different purposes.

The conditions in this section should be more or less self-explanatory. However, I have added a few comments about some of the new clauses and I mention some sentences that are valid in different classes of models. There are many interesting relationships between the various conditions that I will not investigate in this paper. Occasionally, I will mention some connections.

Table 1 contains information about the formal properties of the alethic accessibility relation at a time. In this paper, $\mathfrak{R}$ is treated as a 3-place relation and not as a binary relation as is usually the case ([25], [44], [59], [64] and [118]). Intuitively, this means that the ordinary 2-place alethic accessibility relation is relativised to particular moments in time. 'C' in '$C - aT$' stands for 'condition' and 'a', for 'alethic'. $C - aT$ is called '$C - aT$' because it is a 3-place version of the well-known condition T in ordinary alethic (modal) logic. According to $C - aT$, the alethic accessibility relation $\mathfrak{R}$ is reflexive at every moment in time; according to $C - aD$, $\mathfrak{R}$ is serial at every moment in time, etc. Other conditions in this section are interpreted in a similar way. I will

Table 2: Conditions on the temporal accessibility relation $<$

Condition	Formalisation of condition
$C-PD$	$\forall\tau\exists\tau'\tau'<\tau$
$C-FD$	$\forall\tau\exists\tau'\tau<\tau'$
$C-t4$	$\forall\tau\forall\tau'\forall\tau''((\tau<\tau'\wedge\tau'<\tau'')\rightarrow\tau<\tau'')$
$C-DE$	$\forall\tau\forall\tau'(\tau<\tau'\rightarrow\exists\tau''(\tau<\tau''\wedge\tau''<\tau'))$
$C-FC$	$\forall\tau\forall\tau'\forall\tau''((\tau<\tau'\wedge\tau<\tau'')\rightarrow(\tau'<\tau''\vee\tau'=\tau''\vee\tau''<\tau'))$
$C-PC$	$\forall\tau\forall\tau'\forall\tau''((\tau'<\tau\wedge\tau''<\tau)\rightarrow(\tau'<\tau''\vee\tau'=\tau''\vee\tau''<\tau'))$
$C-C$	$\forall\tau\forall\tau'(\tau<\tau'\vee\tau=\tau'\vee\tau'<\tau)$
$C-UB$	$\forall\tau\forall\tau'\forall\tau''((\tau<\tau'\wedge\tau<\tau'')\rightarrow\exists\tau'''(\tau'<\tau'''\wedge\tau''<\tau'''))$
$C-LB$	$\forall\tau\forall\tau'\forall\tau''((\tau'<\tau\wedge\tau''<\tau)\rightarrow\exists\tau'''(\tau'''<\tau'\wedge\tau'''<\tau''))$

Table 3: Conditions on the deontic accessibility relation $\mathfrak{S}$

Condition	Formalisation of condition
$C-dD$	$\forall\tau\forall\omega\exists\omega'\mathfrak{S}\omega\omega'\tau$
$C-d4$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{S}\omega\omega'\tau\wedge\mathfrak{S}\omega'\omega''\tau)\rightarrow\mathfrak{S}\omega\omega''\tau)$
$C-d5$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{S}\omega\omega'\tau\wedge\mathfrak{S}\omega\omega''\tau)\rightarrow\mathfrak{S}\omega'\omega''\tau)$
$C-\mathbf{O}dT$	$\forall\tau\forall\omega\forall\omega'(\mathfrak{S}\omega\omega'\tau\rightarrow\mathfrak{S}\omega'\omega'\tau)$
$C-\mathbf{O}dB$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{S}\omega\omega'\tau\wedge\mathfrak{S}\omega'\omega''\tau)\rightarrow\mathfrak{S}\omega''\omega'\tau)$

often omit the initial C if it is clear from the context that we are talking about a semantic condition. It is usually binary relations that are called serial, transitive, Euclidean, etc. Nonetheless, I will extend these concepts to 3-place and 4-place relations in this section. If $\mathfrak{A}$ satisfies $C-b4$ (see **Table 4**), we will call $\mathfrak{A}$ transitive, and so on.

The well-known conditions in **Table 2** say something about the formal properties of the temporal relation 'earlier than', $<$. PD stands for 'past D', FD for 'future D', DE for 'dense', FC for 'future convergence', PC for 'past convergence', C for 'comparability', UB for 'upper bounds', and LB for 'lower bounds'. According to $C-PD$, for example, there is no first point in time; according to $C-t4$, time is transitive, etc. The conditions in **Table 2** are often described in various introductions to temporal logic and require no further comments (see, for example, [20], [40], [55], [68], [95], [119], [122] and [112]).

Table 3 includes information about the formal properties of the deontic accessibility relation at a time ('d' stands for 'deontic'). The deontic accessibility relation is usually treated as a binary relation. In this paper, $\mathfrak{S}$ is a 3-place relation. Intuitively, this means that the ordinary binary deontic accessibility relation is relativised to particular moments in time. 2-place versions of the conditions in **Table 3** are discussed in many introductions to deontic logic (see, for example, [7]). According to $C-dD$, the deontic accessibility relation is serial at every moment in time; according to $C-d5$, the deontic accessibility relation is Euclidean at every moment in time, etc.

The condition $C-dT$ ($\forall\tau\forall\omega\mathfrak{S}\omega\omega\tau$), according to which the deontic accessibility

Table 4: Conditions on the boulesic accessibility relation $\mathfrak{A}$

Condition	Formalisation of condition
$C - bD$	$\forall\delta\forall\tau\forall\omega\exists\omega'\mathfrak{A}\delta\omega\omega'\tau$
$C - b4$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{A}\delta\omega\omega'\tau \land \mathfrak{A}\delta\omega'\omega''\tau) \to \mathfrak{A}\delta\omega\omega''\tau)$
$C - b5$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{A}\delta\omega\omega'\tau \land \mathfrak{A}\delta\omega\omega''\tau) \to \mathfrak{A}\delta\omega'\omega''\tau)$
$C - \mathcal{W}bT$	$\forall\delta\forall\tau\forall\omega\forall\omega'(\mathfrak{A}\delta\omega\omega'\tau \to \mathfrak{A}\delta\omega'\omega'\tau)$
$C - \mathcal{W}bB$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{A}\delta\omega\omega'\tau \land \mathfrak{A}\delta\omega'\omega''\tau) \to \mathfrak{A}\delta\omega''\omega'\tau)$

relation is reflexive at every moment in time, is not intuitively plausible. For in every model that satisfies this condition we can show that the schema T (or dT) is valid: $\mathbf{O}A \to A$, which says that everything that is obligatory is true. $C - \mathbf{O}dT$ is similar to $C - dT$, but $C - \mathbf{O}dT$ does not entail that the deontic accessibility relation is reflexive at every moment in time. According to $C - \mathbf{O}dT$, ω' is deontically accessible to itself at a time if ω' is deontically accessible from ω at this time. Likewise, according to the condition $C - dB$ ($\forall\tau\forall\omega\forall\omega'(\mathfrak{S}\omega\omega'\tau \to \mathfrak{S}\omega'\omega\tau)$), the deontic accessibility relation is symmetric at every moment in time. In every model that satisfies this condition, we can show that the schema B (or dB) is valid: $A \to \mathbf{OP}A$, which says that everything that is the case ought to be permitted. Hence, this does not seem to be an intuitively plausible condition. $C - \mathbf{O}dB$ appears to be more reasonable. According to this condition, ω' is accessible from ω'' at τ if ω' is accessible from ω and ω'' is accessible form ω' at τ. $C - \mathbf{O}dT$ does not guarantee that $\mathbf{O}A \to A$ is valid, but it guarantees that T ought to hold. i.e. that $\mathbf{O}(\mathbf{O}A \to A)$ is valid, and $C - \mathbf{O}dB$ does not guarantee that $A \to \mathbf{OP}A$ is valid, but it guarantees that B ought to hold, i.e. that $\mathbf{O}(A \to \mathbf{OP}A)$ is valid.

The conditions in **Table 4**, which deal with the boulesic accessibility relation, are similar to the conditions in **Table 3** (‘b’ stands for ‘boulesic’). However, there are also some important differences; $\mathfrak{S}$ is a 3-place relation, while $\mathfrak{A}$ is a 4-place relation. $C - bD$, for example, says: for every (individual) δ, for every (moment in time) τ and for every (possible world) ω there is a (possible world) ω' such that δ accepts ω' in ω at τ. According to this condition, every individual always accepts at least one possible world at every moment in time, no matter what situation she is in. If a model satisfies this condition, we can show that the following sentence (schema) is valid: $\Pi x(Rx \to \neg(\mathcal{W}_x B \land \mathcal{W}_x \neg B))$, that is, if an individual x is perfectly rational, then it is not the case that x wants it to be the case that B at the same time that x wants it to be the case that not-B. If x wants it to be the case that B and also wants it to be the case that not-B, not all of x's wants can be satisfied. Hence, this principle is intuitively plausible.

Suppose that $\mathcal{M}$ is a model that satisfies $C - UR$, or $C - \Box\mathcal{W}$ and $C - FT$ or $C - FTR$ (see **Table 6** and **Table 16**). Then, if $\mathcal{M}$ satisfies $C - b4$, $\Pi x(Rx \to (\mathcal{W}_x B \to \mathcal{W}_x \mathcal{W}_x B))$ is valid in $\mathcal{M}$; if it satisfies $C - b5$, $\Pi x(Rx \to (\mathcal{A}_x B \to \mathcal{W}_x \mathcal{A}_x B))$ is valid; if it satisfies $C - \mathcal{W}bT$, $\Pi x(Rx \to \mathcal{W}_x(\mathcal{W}_x B \to B))$ is valid; and if it satisfies $C - \mathcal{W}bB$, $\Pi x(Rx \to \mathcal{W}_x(B \to \mathcal{W}_x \mathcal{A}_x B))$ is valid. Note that $C - UR$, or $C - \Box\mathcal{W}$ and $C - FT$ or $C - FTR$ are needed to prove this result since an individual might be perfectly rational in one world-moment pair even though she is not perfectly rational in some other world-moment pair according to our semantics.

Table 5: Alethic deontic interactions: Conditions concerning the relation between $\mathfrak{R}$ and $\mathfrak{S}$

Condition	Formalisation of condition
$C - \Box\mathbf{O}$	$\forall\tau\forall\omega\forall\omega'(\mathfrak{S}\omega\omega'\tau \to \mathfrak{R}\omega\omega'\tau)$
$C - \mathbf{O}\Diamond$	$\forall\tau\forall\omega\exists\omega'(\mathfrak{S}\omega\omega'\tau \wedge \mathfrak{R}\omega\omega'\tau)$
$C - \mathbf{O}\Box\mathbf{O}$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{S}\omega\omega'\tau \wedge \mathfrak{S}\omega'\omega''\tau) \to \mathfrak{R}\omega'\omega''\tau)$
$C - \mathbf{OO}\Diamond$	$\forall\tau\forall\omega\forall\omega'(\mathfrak{S}\omega\omega'\tau \to \exists\omega''(\mathfrak{S}\omega'\omega''\tau \wedge \mathfrak{R}\omega'\omega''\tau))$
$C - da4$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{S}\omega\omega'\tau \wedge \mathfrak{R}\omega'\omega''\tau) \to \mathfrak{R}\omega\omega''\tau)$
$C - da5$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{S}\omega\omega'\tau \wedge \mathfrak{R}\omega\omega''\tau) \to \mathfrak{R}\omega'\omega''\tau)$
$C - ad4$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{R}\omega\omega'\tau \wedge \mathfrak{S}\omega'\omega''\tau) \to \mathfrak{S}\omega\omega''\tau)$
$C - ad5$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{R}\omega\omega'\tau \wedge \mathfrak{S}\omega\omega''\tau) \to \mathfrak{S}\omega'\omega''\tau)$
$C - \mathbf{P}\Box P$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{S}\omega\omega'\tau \wedge \mathfrak{R}\omega\omega''\tau) \to \exists\omega'''(\mathfrak{R}\omega'\omega'''\tau \wedge \mathfrak{S}\omega''\omega'''\tau))$
$C - \mathbf{O}\Box P$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{R}\omega\omega'\tau \wedge \mathfrak{S}\omega'\omega''\tau) \to \exists\omega'''(\mathfrak{S}\omega\omega'''\tau \wedge \mathfrak{R}\omega'''\omega''\tau))$
$C - \Box\mathbf{O}P$	$\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{S}\omega\omega'\tau \wedge \mathfrak{R}\omega'\omega''\tau) \to \exists\omega'''(\mathfrak{R}\omega\omega'''\tau \wedge \mathfrak{S}\omega'''\omega''\tau))$

We observed that $C - dT$ and $C - dB$ are intuitively implausible. Likewise, the conditions $C - bT$ $(\forall\delta\forall\tau\forall\omega\mathfrak{A}\delta\omega\omega\tau)$ and $C - bB$ $(\forall\delta\forall\tau\forall\omega\forall\omega'(\mathfrak{A}\delta\omega\omega'\tau \to \mathfrak{A}\delta\omega'\omega\tau))$ are intuitively problematic. In every model that satisfies $C - bT$, $\Pi x(Rx \to (\mathcal{W}_x B \to B))$ is valid, and in every model that satisfies $C - bB$ (and $C - UR$), $\Pi x(Rx \to (B \to \mathcal{W}_x\mathcal{A}_x B))$ is valid. $C - \mathcal{W}bT$ is weaker than $C - bT$ and $C - \mathcal{W}bB$ is weaker than $C - bB$, and $\Pi x(Rx \to \mathcal{W}_x(\mathcal{W}_x B \to B))$ and $\Pi x(Rx \to \mathcal{W}_x(B \to \mathcal{W}_x\mathcal{A}_x B))$ are intuitively more plausible than $\Pi x(Rx \to (\mathcal{W}_x B \to B))$ and $\Pi x(Rx \to (B \to \mathcal{W}_x\mathcal{A}_x B))$, respectively.

So far, we have considered some formal properties of single accessibility relations. Now, let us turn to some possible connections between two different accessibility relations.

The conditions in **Table 5** are concerned with some possible relations between the alethic and the deontic accessibility relations. In every model that satisfies $C - \Box\mathbf{O}$, $\Box A \to \mathbf{O}A$ (the necessity-ought or must-ought principle) is valid, and in every model that satisfies $C - \mathbf{O}\Diamond$, $\mathbf{O}A \to \Diamond A$ (the ought-possibility or ought-can principle) is valid. $C - \mathbf{O}\Box\mathbf{O}$ is weaker than $C - \Box\mathbf{O}$ and $C - \mathbf{OO}\Diamond$ is weaker than $C - \mathbf{O}\Diamond$. $C - \mathbf{O}\Box\mathbf{O}$ does not guarantee that $\Box A \to \mathbf{O}A$ is valid, but it guarantees that this principle ought to hold, i.e. that $\mathbf{O}(\Box A \to \mathbf{O}A)$ is valid, and $C - \mathbf{OO}\Diamond$ does not guarantee that $\mathbf{O}A \to \Diamond A$ is valid, but it guarantees that this principle ought to hold, i.e. that $\mathbf{O}(\mathbf{O}A \to \Diamond A)$ is valid.

In every model that satisfies $C - da4$, $\Box A \to \mathbf{O}\Box A$ is valid; in every model that satisfies $C - da5$, $\Diamond A \to \mathbf{O}\Diamond A$ is valid; in every model that satisfies $C - ad4$, $\mathbf{O}A \to \Box\mathbf{O}A$ is valid; and in every model that satisfies $C - ad5$, $\mathbf{P}A \to \Box\mathbf{P}A$ is valid.

If a model satisfies $C - \mathbf{P}\Box P$, then $\mathbf{P}\Box A \to \Box\mathbf{P}A$ is valid in this model. If a model satisfies $C - \mathbf{O}\Box P$, then $\mathbf{O}\Box A \to \Box\mathbf{O}A$ is valid in this model. If a model satisfies $C - \Box\mathbf{O}P$, then $\Box\mathbf{O}A \to \mathbf{O}\Box A$ is valid in this model.

'$\Box\mathcal{W}$' in '$C - \Box\mathcal{W}$' stands for 'Must (or Necessity) Want', and '$\mathcal{W}\Diamond$' in '$C - \mathcal{W}\Diamond$' for 'Want Can (or Possibility)' (see **Table 6**). $C - ab4$ (as in 'alethic boulesic 4') is called '$C - ab4$' because it is similar to the well-known alethic (modal) condition $C - 4$ and the alethic deontic condition $C - ad4$, and similarly for $C - ab5$, $C - ba4$ and

Table 6: Alethic boulesic interactions: Conditions concerning the relation between $\mathfrak{R}$ and $\mathfrak{A}$

Condition	Formalisation of condition
$C - \Box\mathcal{W}$	$\forall\delta\forall\tau\forall\omega\forall\omega'(\mathfrak{A}\delta\omega\omega'\tau \rightarrow \mathfrak{R}\omega\omega'\tau)$
$C - \mathcal{W}\Diamond$	$\forall\delta\forall\tau\forall\omega\exists\omega'(\mathfrak{A}\delta\omega\omega'\tau \wedge \mathfrak{R}\omega\omega'\tau)$
$C - \mathcal{W}\Box\mathcal{W}$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{A}\delta\omega\omega'\tau \wedge \mathfrak{A}\delta\omega'\omega''\tau) \rightarrow \mathfrak{R}\omega'\omega''\tau)$
$C - \mathcal{W}\mathcal{W}\Diamond$	$\forall\delta\forall\tau\forall\omega\forall\omega'(\mathfrak{A}\delta\omega\omega'\tau \rightarrow \exists\omega''(\mathfrak{A}\delta\omega'\omega''\tau \wedge \mathfrak{R}\omega'\omega''\tau))$
$C - ba4$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{A}\delta\omega\omega'\tau \wedge \mathfrak{R}\omega'\omega''\tau) \rightarrow \mathfrak{R}\omega\omega''\tau)$
$C - ba5$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{A}\delta\omega\omega'\tau \wedge \mathfrak{R}\omega\omega''\tau) \rightarrow \mathfrak{R}\omega'\omega''\tau)$
$C - ab4$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{R}\omega\omega'\tau \wedge \mathfrak{A}\delta\omega'\omega''\tau) \rightarrow \mathfrak{A}\delta\omega\omega''\tau)$
$C - ab5$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{R}\omega\omega'\tau \wedge \mathfrak{A}\delta\omega\omega''\tau) \rightarrow \mathfrak{A}\delta\omega'\omega''\tau)$
$C - \mathcal{A}\Box P$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{A}\delta\omega\omega'\tau \wedge \mathfrak{R}\omega\omega''\tau) \rightarrow \exists\omega'''(\mathfrak{R}\omega'\omega'''\tau \wedge \mathfrak{A}\delta\omega''\omega'''\tau))$
$C - \mathcal{W}\Box P$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{R}\omega\omega'\tau \wedge \mathfrak{A}\delta\omega'\omega''\tau) \rightarrow \exists\omega'''(\mathfrak{A}\delta\omega\omega'''\tau \wedge \mathfrak{R}\omega'''\omega''\tau))$
$C - \Box\mathcal{W}P$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{A}\delta\omega\omega'\tau \wedge \mathfrak{R}\omega'\omega''\tau) \rightarrow \exists\omega'''(\mathfrak{R}\omega\omega'''\tau \wedge \mathfrak{A}\delta\omega'''\omega''\tau))$

$C - ba5$. '$\mathcal{A}\Box P$' in '$C - \mathcal{A}\Box P$' is an abbreviation of 'Acceptance Must (or Necessity) Permutation'; '$\mathcal{W}\Box P$' and '$\Box\mathcal{W}P$' stand for 'Want Must (or Necessity) Permutation' and 'Must (or Necessity) Want Permutation', respectively.

The conditions in **Table 6** are similar to the conditions in **Table 5**. They are concerned with some possible relationships between the boulesic accessibility relation and the alethic accessibility relation. $C - \Box\mathcal{W}$ says: 'For every (individual) δ, for every (moment in time) τ, for every (possible world) ω and for every (possible world) ω', δ accepts ω' in ω at τ only if ω' is alethically accessible from ω at τ. In other words, if $C - \Box\mathcal{W}$ holds, then it is not the case that δ accepts ω' in ω at τ if ω' is not alethically accessible from ω at τ. In every class of models that satisfies this condition, the following version of the so-called *hypothetical imperative* is valid: $\mathbb{U}\Pi x(Rx \rightarrow ((\mathcal{W}_x A \wedge \Box(A \rightarrow B)) \rightarrow \mathcal{W}_x B))$, which says that if x is perfectly rational, then if x wants A to be the case and it is necessary that A only if B is the case then x wants B to be the case. Hence, this condition is philosophically quite interesting.

$C - \mathcal{W}\Diamond$ is another philosophically interesting condition. According to $C - \mathcal{W}\Diamond$, for every (individual) δ, for every (moment in time) τ, for every (possible world) ω there is a (possible world) ω' such that δ accepts ω' in ω at τ and ω' is alethically accessible from ω at τ. In other words, in every possible world, at every moment in time, δ accepts at least one possible world that is alethically accessible at that time. This condition is similar to condition $C - bD$ (**Table 4**). $C - \mathcal{W}\Diamond$ entails $C - bD$, but $C - bD$ (in itself) does not entail $C - \mathcal{W}\Diamond$. In every class of models that satisfies this condition, the following schema is valid: $\Pi x(Rx \rightarrow (\mathcal{W}_x A \rightarrow \Diamond A))$, that is, if an individual x is perfectly rational, then x wants it to be the case that A only if A is possible. In other words, according to this condition, a perfectly rational individual does not want anything impossible. This is an intuitively plausible principle. If x wants something that is impossible, x's want will inevitably be frustrated.

$C - \mathcal{W}\Box\mathcal{W}$ is weaker than $C - \Box\mathcal{W}$, and $C - \mathcal{W}\mathcal{W}\Diamond$ is weaker than $C - \mathcal{W}\Diamond$. In every model that satisfies $C - \mathcal{W}\Box\mathcal{W}$ (and $C - UR$), $\Pi x(Rx \rightarrow \mathcal{W}_x(\Box A \rightarrow \mathcal{W}_x A))$ is valid, and in every model that satisfies $C - \mathcal{W}\mathcal{W}\Diamond$ (and $C - UR$, or $C - \Box\mathcal{W}$ and $C - FT$

Table 7: Boulesic deontic interactions: Conditions concerning the relation between $\mathfrak{S}$ and $\mathfrak{A}$

Condition	Formalisation of condition
$C-\mathbf{O}\mathcal{W}$	$\forall\delta\forall\tau\forall\omega\forall\omega'(\mathfrak{A}\delta\omega\omega'\tau \to \mathfrak{S}\omega\omega'\tau)$
$C-\mathcal{W}\mathbf{O}$	$\forall\delta\forall\tau\forall\omega\forall\omega'(\mathfrak{S}\omega\omega'\tau \to \mathfrak{A}\delta\omega\omega'\tau)$
$C-\mathbf{O}\mathcal{A}$	$\forall\delta\forall\tau\forall\omega\exists\omega'(\mathfrak{A}\delta\omega\omega'\tau \wedge \mathfrak{S}\omega\omega'\tau)$
$C-bd4$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{A}\delta\omega\omega'\tau \wedge \mathfrak{S}\omega'\omega''\tau) \to \mathfrak{S}\omega\omega''\tau)$
$C-bd5$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{A}\delta\omega\omega'\tau \wedge \mathfrak{S}\omega\omega''\tau) \to \mathfrak{S}\omega'\omega''\tau)$
$C-db4$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{S}\omega\omega'\tau \wedge \mathfrak{A}\delta\omega'\omega''\tau) \to \mathfrak{A}\delta\omega\omega''\tau)$
$C-db5$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{S}\omega\omega'\tau \wedge \mathfrak{A}\delta\omega\omega''\tau) \to \mathfrak{A}\delta\omega'\omega''\tau)$
$C-\mathcal{A}\mathbf{O}P$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{A}\delta\omega\omega'\tau \wedge \mathfrak{S}\omega\omega''\tau) \to \exists\omega'''(\mathfrak{S}\omega'\omega'''\tau \wedge \mathfrak{A}\delta\omega''\omega'''\tau))$
$C-\mathcal{W}\mathbf{O}P$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{S}\omega\omega'\tau \wedge \mathfrak{A}\delta\omega'\omega''\tau) \to \exists\omega'''(\mathfrak{A}\delta\omega\omega'''\tau \wedge \mathfrak{S}\omega'''\omega''\tau))$
$C-\mathbf{O}\mathcal{W}P$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{A}\delta\omega\omega'\tau \wedge \mathfrak{S}\omega'\omega''\tau) \to \exists\omega'''(\mathfrak{S}\omega\omega'''\tau \wedge \mathfrak{A}\delta\omega'''\omega''\tau))$
$C-\mathbf{OO}\mathcal{W}$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{S}\omega\omega'\tau \wedge \mathfrak{A}\delta\omega'\omega''\tau) \to \mathfrak{S}\omega'\omega''\tau)$
$C-\mathbf{O}\mathcal{W}\mathbf{O}$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{S}\omega\omega'\tau \wedge \mathfrak{S}\omega'\omega''\tau) \to \mathfrak{A}\delta\omega'\omega''\tau)$
$C-\mathbf{OO}\mathcal{A}$	$\forall\delta\forall\tau\forall\omega\forall\omega'(\mathfrak{S}\omega\omega'\tau \to \exists\omega''(\mathfrak{A}\delta\omega'\omega''\tau \wedge \mathfrak{S}\omega'\omega''\tau))$
$C-\mathcal{W}\mathbf{O}\mathcal{W}$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{A}\delta\omega\omega'\tau \wedge \mathfrak{A}\delta\omega'\omega''\tau) \to \mathfrak{S}\omega'\omega''\tau)$
$C-\mathcal{W}\mathcal{W}\mathbf{O}$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{A}\delta\omega\omega'\tau \wedge \mathfrak{S}\omega'\omega''\tau) \to \mathfrak{A}\delta\omega'\omega''\tau)$
$C-\mathcal{W}\mathbf{O}\mathcal{A}$	$\forall\delta\forall\tau\forall\omega\forall\omega'(\mathfrak{A}\delta\omega\omega'\tau \to \exists\omega''(\mathfrak{A}\delta\omega'\omega''\tau \wedge \mathfrak{S}\omega'\omega''\tau))$

or $C-FTR$), $\Pi x(Rx \to \mathcal{W}_x(\mathcal{W}_x A \to \Diamond A))$ is valid.

In every model that satisfies $C-ba4$, $\Pi x(Rx \to (\Box A \to \mathcal{W}_x \Box A))$ is valid; in every model that satisfies $C-ba5$, $\Pi x(Rx \to (\Diamond A \to \mathcal{W}_x \Diamond A))$ is valid; in every model that satisfies $C-ab4$ (and $C-UR$, $C-FT$ or $C-FTR$), $\Pi x(Rx \to (\mathcal{W}_x A \to \Box\mathcal{W}_x A))$ is valid; and in every model that satisfies $C-ab5$ (and $C-UR$, $C-FT$ or $C-FTR$), $\Pi x(Rx \to (\mathcal{A}_x B \to \Box\mathcal{A}_x B))$ is valid.

If a model satisfies $C-\mathcal{A}\Box P$ (and $C-UR$, $C-FT$ or $C-FTR$), then $\Pi x(Rx \to (\mathcal{A}_x\Box B \to \Box\mathcal{A}_x B))$ is valid in this model. If a model satisfies $C-\mathcal{W}\Box P$ (and $C-UR$, $C-FT$ or $C-FTR$), then $\Pi x(Rx \to (\mathcal{W}_x \Box A \to \Box\mathcal{W}_x A))$ is valid in this model. If a model satisfies $C-\Box\mathcal{W}P$ (and $C-UR$, $C-FT$ or $C-FTR$), then $\Pi x(Rx \to (\Box\mathcal{W}_x A \to \mathcal{W}_x\Box A))$ is valid in this model.

The conditions in **Table 7** are concerned with some possible relations between the boulesic and the deontic accessibility relations. According to $C-\mathbf{O}\mathcal{W}$, if ω' is boulesically accessible from ω to δ at τ, then ω' is deontically accessible from ω at τ, and according to $C-\mathcal{W}\mathbf{O}$, ω' is boulesically accessible from ω to δ at τ if ω' is deontically accessible from ω at τ. In every $C-\mathbf{O}\mathcal{W}$-model, $\Pi x(Rx \to (\mathbf{O}A \to \mathcal{W}_x A))$ is valid; and in every $C-\mathcal{W}\mathbf{O}$-model, $\Pi x(Rx \to (\mathcal{W}_x A \to \mathbf{O}A))$ is valid. $\Pi x(Rx \to (\mathbf{O}A \to \mathcal{W}_x A))$ is a kind of internalism ('If an individual x is perfectly rational (reasonable or wise), then if it ought to be the case that A then x wants it to be the case that A). $\Pi x(Rx \to (\mathcal{W}_x A \to \mathbf{O}A))$ ('If an individual x is perfectly rational (reasonable or wise), then if x wants it to be the case that A then it ought to be the case that A') is the 'converse' of this proposition. If a model satisfies both $C-\mathbf{O}\mathcal{W}$ and $C-\mathcal{W}\mathbf{O}$, then $\Pi x(Rx \to (\mathbf{O}A \leftrightarrow \mathcal{W}_x A))$ is valid in this model. $\Pi x(Rx \to (\mathbf{O}A \leftrightarrow \mathcal{W}_x A))$ says that if x is perfectly rational (reasonable or wise), then it ought to be the

Table 8: Temporal alethic interactions: Conditions concerning the relation between $\mathfrak{R}$ and $<$

Condition	Formalisation of condition
$C-ASP$	$\forall\tau\forall\tau'\forall\omega\forall\omega'((\tau<\tau'\wedge\mathfrak{R}\omega\omega'\tau')\rightarrow\mathfrak{R}\omega\omega'\tau)$
$C-AR$	$\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau<\tau'\wedge\mathfrak{R}\omega\omega'\tau\wedge\mathfrak{R}\omega'\omega''\tau')\rightarrow\mathfrak{R}\omega\omega''\tau)$

case that A iff x wants it to be the case that A. (See Section 5 for more on this.)

In every $C-\mathbf{O}\mathcal{A}$-model, $\Pi x(Rx\rightarrow(\mathbf{O}B\rightarrow\mathcal{A}_xB))$ is valid. $\Pi x(Rx\rightarrow(\mathbf{O}B\rightarrow\mathcal{A}_xB))$ says that if x is perfectly rational (reasonable or wise), then if it ought to be the case that B then x accepts that it is the case that B.

In every model that satisfies $C-bd4$, $\Pi x(Rx\rightarrow(\mathbf{O}A\rightarrow\mathcal{W}_x\mathbf{O}A))$ is valid; in every model that satisfies $C-bd5$, $\Pi x(Rx\rightarrow(\mathbf{P}A\rightarrow\mathcal{W}_x\mathbf{P}A))$ is valid; in every model that satisfies $C-db4$ (and $C-UR$, or $C-\Box\mathbf{O}$ and $C-FT$ or $C-FTR$), $\Pi x(Rx\rightarrow(\mathcal{W}_xA\rightarrow\mathbf{O}\mathcal{W}_xA))$ is valid; and in every model that satisfies $C-db5$ (and $C-UR$, or $C-\Box\mathbf{O}$ and $C-FT$ or $C-FTR$), $\Pi x(Rx\rightarrow(\mathcal{A}_xB\rightarrow\mathbf{O}\mathcal{A}_xB))$ is valid.

If a model satisfies $C-\mathcal{A}\mathbf{O}P$ (and $C-UR$, or $C-\Box\mathbf{O}$ and $C-FT$ or $C-FTR$), then $\Pi x(Rx\rightarrow(\mathcal{A}_x\mathbf{O}B\rightarrow\mathbf{O}\mathcal{A}_xB))$ is valid in this model. If a model satisfies $C-\mathcal{W}\mathbf{O}P$ (and $C-UR$, or $C-\Box\mathbf{O}$ and $C-FT$ or $C-FTR$), then $\Pi x(Rx\rightarrow(\mathcal{W}_x\mathbf{O}A\rightarrow\mathbf{O}\mathcal{W}_xA))$ is valid in this model. If a model satisfies $C-\mathbf{O}\mathcal{W}P$ (and $C-UR$, or $C-\Box\mathbf{O}$ and $C-FT$ or $C-FTR$), then $\Pi x(Rx\rightarrow(\mathbf{O}\mathcal{W}_xA\rightarrow\mathcal{W}_x\mathbf{O}A))$ is valid in this model.

$C-\mathbf{OO}\mathcal{W}$ is weaker than $C-\mathbf{O}\mathcal{W}$, $C-\mathbf{O}\mathcal{W}\mathbf{O}$ is weaker than $C-\mathcal{W}\mathbf{O}$, and so on for $C-\mathbf{OO}\mathcal{A}$, $C-\mathcal{W}\mathbf{O}\mathcal{W}$, $C-\mathcal{WW}\mathbf{O}$, and $C-\mathcal{W}\mathbf{O}\mathcal{A}$.

In every model that satisfies $C-\mathbf{OO}\mathcal{W}$ (and $C-UR$, or $C-\Box\mathbf{O}$ and $C-FT$ or $C-FTR$), $\Pi x(Rx\rightarrow\mathbf{O}(\mathbf{O}A\rightarrow\mathcal{W}_xA))$ is valid; in every model that satisfies $C-\mathbf{O}\mathcal{W}\mathbf{O}$ (and $C-UR$, or $C-\Box\mathbf{O}$ and $C-FT$ or $C-FTR$), $\Pi x(Rx\rightarrow\mathbf{O}(\mathcal{W}_xA\rightarrow\mathbf{O}A))$ is valid; in every model that satisfies $C-\mathbf{OO}\mathcal{A}$ (and $C-UR$, or $C-\Box\mathbf{O}$ and $C-FT$ or $C-FTR$), $\Pi x(Rx\rightarrow\mathbf{O}(\mathbf{O}B\rightarrow\mathcal{A}_xB))$ is valid; in every model that satisfies $C-\mathcal{W}\mathbf{O}\mathcal{W}$ (and $C-UR$, or $C-\Box\mathcal{W}$ and $C-FT$ or $C-FTR$), $\Pi x(Rx\rightarrow\mathcal{W}_x(\mathbf{O}A\rightarrow\mathcal{W}_xA))$ is valid; in every model that satisfies $C-\mathcal{WW}\mathbf{O}$ (and $C-UR$, or $C-\Box\mathcal{W}$ and $C-FT$ or $C-FTR$), $\Pi x(Rx\rightarrow\mathcal{W}_x(\mathcal{W}_xA\rightarrow\mathbf{O}A))$ is valid; and in every model that satisfies $C-\mathcal{W}\mathbf{O}\mathcal{A}$ (and $C-UR$, or $C-\Box\mathcal{W}$ and $C-FT$ or $C-FTR$), $\Pi x(Rx\rightarrow\mathcal{W}_x(\mathbf{O}B\rightarrow\mathcal{A}_xB))$ is valid.

The conditions in **Table 8** are concerned with some possible relations between $\mathfrak{R}$ and $<$. In the conditions in this table, ASP stands for 'alethic shared past' and AR for 'alethic ramification'.

According to $C-ASP$, it is true that if a world ω' is alethically accessible from a world ω at time τ', then ω' is alethically accessible from ω at every moment τ that is earlier than τ'. This condition is plausible if we model reality as a tree-like structure that branches towards the future and not as a set of entirely unconnected possible worlds and moments in time. Then we can think of the possible worlds in W as possible histories of one and the same world (reality) rather than as distinct worlds.

Note that $C-AR$ follows from $C-ASP$ and $C-a4$. $C-AR$ is, therefore, also plausible if we model the world as a tree-like structure.

If a model satisfies $C-ASP$, we can show that the following sentences are valid:

Table 9: Temporal deontic interactions: Conditions concerning the relation between $\mathfrak{S}$ and $<$

Condition	Formalisation of condition
$C-\mathbf{O}\mathbb{G}dT$	$\forall\tau\forall\tau'\forall\omega\forall\omega'((\tau<\tau' \land \mathfrak{S}\omega\omega'\tau) \rightarrow \mathfrak{S}\omega'\omega'\tau')$
$C-\mathbf{O}\mathbb{G}dB$	$\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau<\tau' \land \mathfrak{S}\omega\omega'\tau \land \mathfrak{S}\omega'\omega''\tau') \rightarrow \mathfrak{S}\omega''\omega'\tau')$
$C-DR$	$\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau<\tau' \land \mathfrak{S}\omega\omega'\tau \land \mathfrak{S}\omega'\omega''\tau') \rightarrow \mathfrak{S}\omega\omega''\tau)$

Table 10: Temporal boulesic interactions: Conditions concerning the relation between $\mathfrak{A}$ and $<$

Condition	Formalisation of condition
$C-\mathcal{W}\mathbb{G}bT$	$\forall\delta\forall\tau\forall\tau'\forall\omega\forall\omega'((\tau<\tau' \land \mathfrak{A}\delta\omega\omega'\tau) \rightarrow \mathfrak{A}\delta\omega'\omega'\tau')$
$C-\mathcal{W}\mathbb{G}bB$	$\forall\delta\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau<\tau' \land \mathfrak{A}\delta\omega\omega'\tau \land \mathfrak{A}\delta\omega'\omega''\tau') \rightarrow \mathfrak{A}\delta\omega''\omega'\tau')$
$C-BR$	$\forall\delta\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau<\tau' \land \mathfrak{A}\delta\omega\omega'\tau \land \mathfrak{A}\delta\omega'\omega''\tau') \rightarrow \mathfrak{A}\delta\omega\omega''\tau)$

$\mathbb{H}\Box A \rightarrow \Box\mathbb{H}A$, $\mathbb{P}\Box A \rightarrow \Box\mathbb{P}A$, $\Box\mathbb{G}A \rightarrow \mathbb{G}\Box A$ and $\Box A \rightarrow \mathbb{G}\Box\mathbb{P}A$.

If a model satisfies $C-AR$, we can show that $\Box\mathbb{G}A \rightarrow \Box\mathbb{G}\Box A$ is valid.

The conditions in **Table 9** are concerned with some possible relations between the temporal and the deontic accessibility relations.

In every $C-\mathbf{O}\mathbb{G}dT$-model, $\mathbf{O}\mathbb{G}(\mathbf{O}A \rightarrow A)$ is valid, and in every $C-\mathbf{O}\mathbb{G}dB$-model, $\mathbf{O}\mathbb{G}(A \rightarrow \mathbf{OP}A)$ is valid. If a model satisfies $C-DR$ ('deontic ramification'), $\mathbf{O}\mathbb{G}A \rightarrow \mathbf{O}\mathbb{G}\mathbf{O}A$ is valid in this model.

$C-\mathbf{O}\mathbb{G}dT$ is weaker than $C-dT$ and $C-\mathbf{O}\mathbb{G}dB$ is weaker than $C-dB$; if a model satisfies $C-dT$, it also satisfies $C-\mathbf{O}\mathbb{G}dT$, and if a model satisfies $C-dB$, then it satisfies $C-\mathbf{O}\mathbb{G}dB$. However, we have already observed that $C-dT$ and $C-dB$ are intuitively implausible. $\mathbf{O}\mathbb{G}(\mathbf{O}A \rightarrow A)$ says that it ought to be that it is always going to be that if A ought to be then A is true; in other words, it says that it ought to be that it is always going to be the case that dT is true (not that dT is in fact true), and this seems to be intuitively much more plausible than dT itself. $\mathbf{O}\mathbb{G}(A \rightarrow \mathbf{OP}A)$ says that it ought to be that it is always going to be that if A is true then it ought to be permitted that A. In other words, this sentence says that it ought to be that it is always going to be that dB is true. This is intuitively more plausible than dB itself.

Note that if a model satisfies both $C-\mathbf{O}\mathbb{G}dT$ and $C-\mathbf{O}dT$, then $\mathbf{O}\underline{\mathbb{G}}(\mathbf{O}A \rightarrow A)$ is valid in this model, that is, then it is true that it ought to be that it is and that it is always going to be the case that if it ought to be the case that A then A. $\mathbf{O}\underline{\mathbb{G}}(\mathbf{O}A \rightarrow A)$ is by definition equivalent with $\mathbf{O}((\mathbf{O}A \rightarrow A) \land \mathbb{G}(\mathbf{O}A \rightarrow A))$. Likewise, if a model satisfies both $C-\mathbf{O}\mathbb{G}dB$ and $C-\mathbf{O}dB$, then $\mathbf{O}\underline{\mathbb{G}}(A \rightarrow \mathbf{OP}A)$ is valid in this model. $\mathbf{O}\underline{\mathbb{G}}(A \rightarrow \mathbf{OP}A)$ is by definition equivalent with $\mathbf{O}((A \rightarrow \mathbf{OP}A) \land \mathbb{G}(A \rightarrow \mathbf{OP}A))$.

The conditions in **Table 10** are similar to the conditions in **Table 9**. They are concerned with some possible connections between $\mathfrak{A}$ and $<$. If a model satisfies $C-\mathcal{W}\mathbb{G}bT$ (and $C-UR$), $\Pi x(Rx \rightarrow \mathcal{W}_x\mathbb{G}(\mathcal{W}_xB \rightarrow B))$ is valid in this model, and if a model satisfies $C-\mathcal{W}\mathbb{G}bB$ (and $C-UR$), $\Pi x(Rx \rightarrow \mathcal{W}_x\mathbb{G}(B \rightarrow \mathcal{W}_x\mathcal{A}_xB))$ is valid in

Table 11: Alethic boulesic deontic interactions: Conditions concerning the relation between $\mathfrak{R}$, $\mathfrak{A}$ and $\mathfrak{S}$

Condition	Formalisation of condition
$C - \mathbf{O} \Box \mathcal{W}$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{S}\omega\omega'\tau \land \mathfrak{A}\delta\omega'\omega''\tau) \to \mathfrak{R}\omega'\omega''\tau)$
$C - \mathcal{W} \Box \mathbf{O}$	$\forall\delta\forall\tau\forall\omega\forall\omega'\forall\omega''((\mathfrak{A}\delta\omega\omega'\tau \land \mathfrak{S}\omega'\omega''\tau) \to \mathfrak{R}\omega'\omega''\tau)$
$C - \mathbf{O}\mathcal{W}\Diamond$	$\forall\delta\forall\tau\forall\omega\forall\omega'(\mathfrak{S}\omega\omega'\tau \to \exists\omega''(\mathfrak{A}\delta\omega'\omega''\tau \land \mathfrak{R}\omega'\omega''\tau))$
$C - \mathcal{W}\mathbf{O}\Diamond$	$\forall\delta\forall\tau\forall\omega\forall\omega'(\mathfrak{A}\delta\omega\omega'\tau \to \exists\omega''(\mathfrak{S}\omega'\omega''\tau \land \mathfrak{R}\omega'\omega''\tau))$

this model. $\Pi x(Rx \to (\mathcal{W}_x\mathbb{G}B \to \mathcal{W}_x\mathbb{G}\mathcal{W}_xB))$ is valid in every model that satisfies $C - BR$ (‘boulesic ramification’) (and $C - UR$).

Note that if a model satisfies both $C - \mathcal{W}\mathbb{G}bT$ and $C - \mathcal{W}bT$ (and $C - UR$), then $\Pi x(Rx \to \mathcal{W}_x\underline{\mathbb{G}}(\mathcal{W}_xB \to B))$ is valid in this model, that is, then it is true that if x is perfectly rational, then x wants it to be that it is and that it is always going to be the case that if x wants it to be the case that B then B. $\Pi x(Rx \to \mathcal{W}_x\underline{\mathbb{G}}(\mathcal{W}_xB \to B))$ is by definition equivalent with $\Pi x(Rx \to \mathcal{W}_x((\mathcal{W}_xB \to B) \land \mathbb{G}(\mathcal{W}_xB \to B)))$. Likewise, if a model satisfies both $C - \mathcal{W}\mathbb{G}bB$ and $C - \mathcal{W}bB$ (and $C - UR$), then $\Pi x(Rx \to \mathcal{W}_x\underline{\mathbb{G}}(B \to \mathcal{W}_x\mathcal{A}_xB))$ is valid in this model. $\Pi x(Rx \to \mathcal{W}_x\underline{\mathbb{G}}(B \to \mathcal{W}_x\mathcal{A}_xB))$ is by definition equivalent with $\Pi x(Rx \to \mathcal{W}_x((B \to \mathcal{W}_x\mathcal{A}_xB) \land \mathbb{G}(B \to \mathcal{W}_x\mathcal{A}_xB)))$.

So far, we have considered some formal properties of single accessibility relations and some possible interactions between two different accessibility relations. Now, let us investigate some possible connections that involve three different accessibility relations.

The conditions in **Table 11** concern some possible interactions between $\mathfrak{R}$, $\mathfrak{A}$ and $\mathfrak{S}$.

$C - \Box\mathcal{W}$ is stronger than $C - \mathbf{O} \Box \mathcal{W}$, $C - \Box\mathbf{O}$ is stronger than $C - \mathcal{W} \Box \mathbf{O}$, $C - \mathcal{W}\Diamond$ is stronger than $C - \mathbf{O}\mathcal{W}\Diamond$, and $C - \mathbf{O}\Diamond$ is stronger than $C - \mathcal{W}\mathbf{O}\Diamond$. Every sentence that is valid in a $C - \mathbf{O} \Box \mathcal{W}$-model is therefore also valid in a $C - \Box\mathcal{W}$-model, etc.

If a model satisfies $C - \mathbf{O} \Box \mathcal{W}$ (and $T - UR$), then $\Pi x(Rx \to \mathbf{O}(\Box A \to \mathcal{W}_xA))$ is valid in this model. If a model satisfies $C - \mathcal{W} \Box \mathbf{O}$ (and $T - UR$), then $\Pi x(Rx \to \mathcal{W}_x(\Box A \to \mathbf{O}A))$ is valid in this model. $\Pi x(Rx \to \mathbf{O}(\mathcal{W}_xA \to \Diamond A))$ is valid in every model that satisfies $C - \mathbf{O}\mathcal{W}\Diamond$ (and $T - UR$), and $\Pi x(Rx \to \mathcal{W}_x(\mathbf{O}A \to \Diamond A))$ is valid in every model that satisfies $C - \mathcal{W}\mathbf{O}\Diamond$ (and $T - UR$).

If it is reasonable to accept $C - \mathcal{W}\Diamond$, then it is also reasonable to accept $C - \mathbf{O}\mathcal{W}\Diamond$ since the latter is derivable from the former. However, some might think that $C - \mathcal{W}\Diamond$ is too strong. Such an individual might still believe that $C - \mathbf{O}\mathcal{W}\Diamond$ is reasonable. According to $\Pi x(Rx \to (\mathcal{W}_xA \to \Diamond A))$, every perfectly rational individual wants something only if it is possible. However, according to $\Pi x(Rx \to \mathbf{O}(\mathcal{W}_xA \to \Diamond A))$, this is not necessarily the case. Even if this sentence is true, it is possible that someone that is perfectly rational wants something that is impossible. But it is true that if someone is perfecty rational, then it ought to be that she wants something only if it is possible according to this formula. Similar remarks apply to the other conditions in **Table 11**. In this paper, I will not try to decide whether or not this position is plausible, but it is clearly interesting enough to be worth mentioning.

Table 12: Temporal boulesic deontic interactions: Conditions concerning the relation between $\mathfrak{A}$, $\mathfrak{S}$ and $<$

Condition	Formalisation of condition
$C-\mathbf{O}\mathbb{G}bT$	$\forall\delta\forall\tau\forall\tau'\forall\omega\forall\omega'((\tau<\tau'\wedge\mathfrak{S}\omega\omega'\tau)\rightarrow\mathfrak{A}\delta\omega'\omega'\tau')$
$C-\mathbf{O}\mathbb{G}bB$	$\forall\delta\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau<\tau'\wedge\mathfrak{S}\omega\omega'\tau\wedge\mathfrak{A}\delta\omega'\omega''\tau')\rightarrow\mathfrak{A}\delta\omega''\omega'\tau')$
$C-\mathbf{O}\mathbb{G}\mathbf{O}\mathcal{W}$	$\forall\delta\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau<\tau'\wedge\mathfrak{S}\omega\omega'\tau\wedge\mathfrak{A}\delta\omega'\omega''\tau')\rightarrow\mathfrak{S}\omega'\omega''\tau')$
$C-\mathbf{O}\mathbb{G}\mathcal{W}\mathbf{O}$	$\forall\delta\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau<\tau'\wedge\mathfrak{S}\omega\omega'\tau\wedge\mathfrak{S}\omega'\omega''\tau')\rightarrow\mathfrak{A}\delta\omega'\omega''\tau')$
$C-\mathbf{O}\mathbb{G}\mathbf{O}\mathcal{A}$	$\forall\delta\forall\tau\forall\tau'\forall\omega\forall\omega'((\tau<\tau'\wedge\mathfrak{S}\omega\omega'\tau)\rightarrow\exists\omega''(\mathfrak{A}\delta\omega'\omega''\tau'\wedge\mathfrak{S}\omega'\omega''\tau'))$
$C-\mathcal{W}\mathbb{G}dT$	$\forall\delta\forall\tau\forall\tau'\forall\omega\forall\omega'((\tau<\tau'\wedge\mathfrak{A}\delta\omega\omega'\tau)\rightarrow\mathfrak{S}\omega'\omega'\tau')$
$C-\mathcal{W}\mathbb{G}dB$	$\forall\delta\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau<\tau'\wedge\mathfrak{A}\delta\omega\omega'\tau\wedge\mathfrak{S}\omega'\omega''\tau')\rightarrow\mathfrak{S}\omega''\omega'\tau')$
$C-\mathcal{W}\mathbb{G}\mathbf{O}\mathcal{W}$	$\forall\delta\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau<\tau'\wedge\mathfrak{A}\delta\omega\omega'\tau\wedge\mathfrak{A}\delta\omega'\omega''\tau')\rightarrow\mathfrak{S}\omega'\omega''\tau')$
$C-\mathcal{W}\mathbb{G}\mathcal{W}\mathbf{O}$	$\forall\delta\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau<\tau'\wedge\mathfrak{A}\delta\omega\omega'\tau\wedge\mathfrak{S}\omega'\omega''\tau')\rightarrow\mathfrak{A}\delta\omega'\omega''\tau')$
$C-\mathcal{W}\mathbb{G}\mathbf{O}\mathcal{A}$	$\forall\delta\forall\tau\forall\tau'\forall\omega\forall\omega'((\tau<\tau'\wedge\mathfrak{A}\delta\omega\omega'\tau)\rightarrow\exists\omega''(\mathfrak{A}\delta\omega'\omega''\tau'\wedge\mathfrak{S}\omega'\omega''\tau'))$

The conditions in **Table 12** concern some possible relationships between $\mathfrak{A}$, $\mathfrak{S}$ and $<$.

If a model satisfies $C-\mathbf{O}\mathbb{G}bT$ (and $C-UR$), then $\Pi x(Rx\rightarrow\mathbf{O}\mathbb{G}(\mathcal{W}_xA\rightarrow A))$ is valid in this model, and if a model satisfies $C-\mathbf{O}\mathbb{G}bB$ (and $C-UR$), then $\Pi x(Rx\rightarrow\mathbf{O}\mathbb{G}(B\rightarrow\mathcal{W}_x\mathcal{A}_xB))$ is valid in this model. $\Pi x(Rx\rightarrow\mathbf{O}\mathbb{G}(\mathbf{O}A\rightarrow\mathcal{W}_xA))$ is valid in every model that satisfies $C-\mathbf{O}\mathbb{G}\mathbf{O}\mathcal{W}$ (and $C-UR$); $\Pi x(Rx\rightarrow\mathbf{O}\mathbb{G}(\mathcal{W}_xA\rightarrow\mathbf{O}A))$ is valid in every model that satisfies $C-\mathbf{O}\mathbb{G}\mathcal{W}\mathbf{O}$ (and $C-UR$); and $\Pi x(Rx\rightarrow\mathbf{O}\mathbb{G}(\mathbf{O}B\rightarrow\mathcal{A}_xB))$ is valid in every model that satisfies $C-\mathbf{O}\mathbb{G}\mathbf{O}\mathcal{A}$ (and $C-UR$). If a model satisfies $C-\mathcal{W}\mathbb{G}dT$ (and $C-UR$), $\Pi x(Rx\rightarrow\mathcal{W}_x\mathbb{G}(\mathbf{O}A\rightarrow A))$ is valid in this model, and if a model satisfies $C-\mathcal{W}\mathbb{G}dB$ (and $C-UR$), $\Pi x(Rx\rightarrow\mathcal{W}_x\mathbb{G}(A\rightarrow\mathbf{OP}A))$ is valid in this model. $\Pi x(Rx\rightarrow\mathcal{W}_x\mathbb{G}(\mathbf{O}A\rightarrow\mathcal{W}_xA))$ is valid in every model that satisfies $C-\mathcal{W}\mathbb{G}\mathbf{O}\mathcal{W}$ (and $C-UR$); $\Pi x(Rx\rightarrow\mathcal{W}_x\mathbb{G}(\mathcal{W}_xA\rightarrow\mathbf{O}A))$ is valid in every model that satisfies $C-\mathcal{W}\mathbb{G}\mathcal{W}\mathbf{O}$ (and $C-UR$); and $\Pi x(Rx\rightarrow\mathcal{W}_x\mathbb{G}(\mathbf{O}B\rightarrow\mathcal{A}_xB))$ is valid in every model that satisfies $C-\mathcal{W}\mathbb{G}\mathbf{O}\mathcal{A}$ (and $C-UR$).

$C-\mathbf{O}\mathbb{G}\mathbf{O}\mathcal{W}$ is weaker than $C-\mathbf{O}\mathcal{W}$, $C-\mathbf{O}\mathbb{G}\mathcal{W}\mathbf{O}$ is weaker than $C-\mathcal{W}\mathbf{O}$, $C-\mathbf{O}\mathbb{G}\mathbf{O}\mathcal{A}$ is weaker than $C-\mathbf{O}\mathcal{A}$, $C-\mathcal{W}\mathbb{G}\mathbf{O}\mathcal{W}$ is weaker than $C-\mathbf{O}\mathcal{W}$, $C-\mathcal{W}\mathbb{G}\mathcal{W}\mathbf{O}$ is weaker than $C-\mathcal{W}\mathbf{O}$, and $C-\mathcal{W}\mathbb{G}\mathbf{O}\mathcal{A}$ is weaker than $C-\mathbf{O}\mathcal{A}$. Every sentence that is valid in a $C-\mathbf{O}\mathbb{G}\mathbf{O}\mathcal{W}$-model is therefore also valid in a $C-\mathbf{O}\mathcal{W}$-model, etc.

Suppose a model satisfies $C-\mathbf{O}\mathbb{G}bT$ and $C-\mathbf{O}bT$ (and $C-UR$). Then $\Pi x(Rx\rightarrow\mathbf{O}\underline{\mathbb{G}}(\mathcal{W}_xA\rightarrow A))$ is valid in this model. Suppose a model satisfies $C-\mathbf{O}\mathbb{G}bB$ and $C-\mathbf{O}bB$ (and $C-UR$). Then $\Pi x(Rx\rightarrow\mathbf{O}\underline{\mathbb{G}}(B\rightarrow\mathcal{W}_x\mathcal{A}_xB))$ is valid in this model. $\Pi x(Rx\rightarrow\mathbf{O}\underline{\mathbb{G}}(\mathbf{O}A\rightarrow\mathcal{W}_xA))$ is valid in every model that satisfies $C-\mathbf{O}\mathbb{G}\mathbf{O}\mathcal{W}$ and $C-\mathbf{OO}\mathcal{W}$ (and $C-UR$); $\Pi x(Rx\rightarrow\mathbf{O}\underline{\mathbb{G}}(\mathcal{W}_xA\rightarrow\mathbf{O}A))$ is valid in every model that satisfies $C-\mathbf{O}\mathbb{G}\mathcal{W}\mathbf{O}$ and $C-\mathbf{O}\mathcal{W}\mathbf{O}$ (and $C-UR$); and $\Pi x(Rx\rightarrow\mathbf{O}\underline{\mathbb{G}}(\mathbf{O}B\rightarrow\mathcal{A}_xB))$ is valid in every model that satisfies $C-\mathbf{O}\mathbb{G}\mathbf{O}\mathcal{A}$ and $C-\mathbf{OO}\mathcal{A}$ (and $C-UR$). If a model satisfies $C-\mathcal{W}\mathbb{G}dT$ and $C-\mathcal{W}dT$ (and $C-UR$), $\Pi x(Rx\rightarrow\mathcal{W}_x\underline{\mathbb{G}}(\mathbf{O}A\rightarrow A))$ is valid in this model, and if a model satisfies $C-\mathcal{W}\mathbb{G}dB$ and $C-\mathcal{W}dB$ (and $C-UR$), $\Pi x(Rx\rightarrow\mathcal{W}_x\underline{\mathbb{G}}(A\rightarrow\mathbf{OP}A))$ is valid in this model. $\Pi x(Rx\rightarrow\mathcal{W}_x\underline{\mathbb{G}}(\mathbf{O}A\rightarrow\mathcal{W}_xA))$ is valid in every model that satisfies $C-\mathcal{W}\mathbb{G}\mathbf{O}\mathcal{W}$ and $C-\mathcal{W}\mathbf{O}\mathcal{W}$ (and $C-UR$);

Table 13: Temporal alethic deontic interactions: Conditions concerning the relation between $\mathfrak{R}$, $\mathfrak{S}$ and $<$

Condition	Formalisation of condition
$C - \mathbf{O}\mathbb{G} \Box \mathbf{O}$	$\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau < \tau' \wedge \mathfrak{S}\omega\omega'\tau \wedge \mathfrak{S}\omega'\omega''\tau') \rightarrow \mathfrak{R}\omega'\omega''\tau')$
$C - \mathbf{O}\mathbb{G}\mathbf{O}\Diamond$	$\forall\tau\forall\tau'\forall\omega\forall\omega'((\tau < \tau' \wedge \mathfrak{S}\omega\omega'\tau) \rightarrow \exists\omega''(\mathfrak{S}\omega'\omega''\tau' \wedge \mathfrak{R}\omega'\omega''\tau'))$

Table 14: Temporal alethic boulesic interactions: Conditions concerning the relation between $\mathfrak{R}$, $\mathfrak{A}$ and $<$

Condition	Formalisation of condition
$C - \mathcal{W}\mathbb{G} \Box \mathcal{W}$	$\forall\delta\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau < \tau' \wedge \mathfrak{A}\delta\omega\omega'\tau \wedge \mathfrak{A}\delta\omega'\omega''\tau') \rightarrow \mathfrak{R}\omega'\omega''\tau')$
$C - \mathcal{W}\mathbb{G}\mathcal{W}\Diamond$	$\forall\delta\forall\tau\forall\tau'\forall\omega\forall\omega'((\tau < \tau' \wedge \mathfrak{A}\delta\omega\omega'\tau) \rightarrow \exists\omega''(\mathfrak{A}\delta\omega'\omega''\tau' \wedge \mathfrak{R}\omega'\omega''\tau'))$

$\Pi x(Rx \rightarrow \mathcal{W}_x\underline{\mathbb{G}}(\mathcal{W}_xA \rightarrow \mathbf{O}A))$ is valid in every model that satisfies $C - \mathcal{W}\mathbb{G}\mathcal{W}\mathbf{O}$ and $C - \mathcal{W}\mathcal{W}\mathbf{O}$ (and $C - UR$); and $\Pi x(Rx \rightarrow \mathcal{W}_x\underline{\mathbb{G}}(\mathbf{O}B \rightarrow \mathcal{A}_xB))$ is valid in every model that satisfies $C - \mathcal{W}\mathbb{G}\mathbf{O}\mathcal{A}$ and $C - \mathcal{W}\mathbf{O}\mathcal{A}$ (and $C - UR$).

The conditions in **Table 13** are concerned with some possible interactions between $\mathfrak{R}$, $\mathfrak{S}$ and $<$. In every $C - \mathbf{O}\mathbb{G} \Box \mathbf{O}$-model, $\mathbf{O}\mathbb{G}(\Box A \rightarrow \mathbf{O}A)$ is valid, and in every $C - \mathbf{O}\mathbb{G}\mathbf{O}\Diamond$-model, $\mathbf{O}\mathbb{G}(\mathbf{O}A \rightarrow \Diamond A)$ is valid. If a model satisfies $C - \mathbf{O}\mathbb{G} \Box \mathbf{O}$ and $C - \mathbf{O} \Box \mathbf{O}$, $\mathbf{O}\underline{\mathbb{G}}(\Box A \rightarrow \mathbf{O}A)$ is valid in this model, and if a model satisfies $C - \mathbf{O}\mathbb{G}\mathbf{O}\Diamond$ and $C - \mathbf{O}\mathbf{O}\Diamond$, $\mathbf{O}\underline{\mathbb{G}}(\mathbf{O}A \rightarrow \Diamond A)$ is valid in this model.

The conditions in **Table 14** are concerned with some possible interactions between $\mathfrak{R}$, $\mathfrak{A}$ and $<$. If a model satisfies $C - \mathcal{W}\mathbb{G} \Box \mathcal{W}$ (and $C - UR$), $\Pi x(Rx \rightarrow \mathcal{W}_x\mathbb{G}(\Box A \rightarrow \mathcal{W}_xA))$ is valid in this model; and if a model satisfies $C - \mathcal{W}\mathbb{G}\mathcal{W}\Diamond$ (and $C - UR$), $\Pi x(Rx \rightarrow \mathcal{W}_x\mathbb{G}(\mathcal{W}_xA \rightarrow \Diamond A))$ is valid in this model. If a model satisfies $C - \mathcal{W}\mathbb{G} \Box \mathcal{W}$ and $C - \mathcal{W} \Box \mathcal{W}$ (and $C - UR$), $\Pi x(Rx \rightarrow \mathcal{W}_x\underline{\mathbb{G}}(\Box A \rightarrow \mathcal{W}_xA))$ is valid in this model, and if it satisfies $C - \mathcal{W}\mathbb{G}\mathcal{W}\Diamond$ and $C - \mathcal{W}\mathcal{W}\Diamond$ (and $C - UR$), $\Pi x(Rx \rightarrow \mathcal{W}_x\underline{\mathbb{G}}(\mathcal{W}_xA \rightarrow \Diamond A))$ is valid in this model.

Finally, let us consider some possible interactions between all four accessibility relations (see **Table 15**).

If a model satisfies $C - \mathbf{O}\mathbb{G} \Box \mathcal{W}$ (and $C - UR$), $\Pi x(Rx \rightarrow \mathbf{O}\mathbb{G}(\Box A \rightarrow \mathcal{W}_xA))$ is valid

Table 15: Temporal alethic boulesic deontic interactions: Conditions concerning the relation between $\mathfrak{R}$, $\mathfrak{A}$, $\mathfrak{S}$ and $<$

Condition	Formalisation of condition
$C - \mathbf{O}\mathbb{G} \Box \mathcal{W}$	$\forall\delta\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau < \tau' \wedge \mathfrak{S}\omega\omega'\tau \wedge \mathfrak{A}\delta\omega'\omega''\tau') \rightarrow \mathfrak{R}\omega'\omega''\tau')$
$C - \mathbf{O}\mathbb{G}\mathcal{W}\Diamond$	$\forall\delta\forall\tau\forall\tau'\forall\omega\forall\omega'((\tau < \tau' \wedge \mathfrak{S}\omega\omega'\tau) \rightarrow \exists\omega''(\mathfrak{A}\delta\omega'\omega''\tau' \wedge \mathfrak{R}\omega'\omega''\tau'))$
$C - \mathcal{W}\mathbb{G} \Box \mathbf{O}$	$\forall\delta\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau < \tau' \wedge \mathfrak{A}\delta\omega\omega'\tau \wedge \mathfrak{S}\omega'\omega''\tau') \rightarrow \mathfrak{R}\omega'\omega''\tau')$
$C - \mathcal{W}\mathbb{G}\mathbf{O}\Diamond$	$\forall\delta\forall\tau\forall\tau'\forall\omega\forall\omega'\forall\omega''((\tau < \tau' \wedge \mathfrak{A}\delta\omega\omega'\tau) \rightarrow \exists\omega''(\mathfrak{S}\omega'\omega''\tau' \wedge \mathfrak{R}\omega'\omega''\tau'))$

Table 16: Conditions on the valuation function v in a model

Condition	Formalisation of condition
$C-FT$	If $\mathfrak{R}\omega_1\omega_2\tau$ and A is an atomic sentence (or a sentence of the form $\mathcal{W}_cB$, $\mathcal{A}_cB$, $\mathcal{R}_cB$, $\mathcal{I}_cB$ or $\mathcal{N}_cB$, given that $\neg Rc$ is true in ω_1 at τ) that is true in ω_1 at τ, then A is true in ω_2 at τ.
$C-BT$	If $\mathfrak{R}\omega_1\omega_2\tau$ and A is an atomic sentence (or a sentence of the form $\mathcal{W}_cB$, $\mathcal{A}_cB$, $\mathcal{R}_cB$, $\mathcal{I}_cB$ or $\mathcal{N}_cB$, given that $\neg Rc$ is true in ω_2 at τ) that is true in ω_2 at τ, then A is true in ω_1 at τ.
$C-FTR$	If $\mathfrak{R}\omega_1\omega_2\tau$ and Rc is true in ω_1 at τ, then Rc is true in ω_2 at τ (for any c).
$C-UR$	If Rc is true in ω_1 at τ_1, then Rc is true in ω_2 at τ_2 (for any c).

in this model; if it satisfies $C-\mathbf{O}\mathbb{G}\mathcal{W}\Diamond$ (and $C-UR$), $\Pi x(Rx \to \mathbf{O}\mathbb{G}(\mathcal{W}_xA \to \Diamond A))$ is valid in this model; if it satisfies $C-\mathcal{W}\mathbb{G}\Box\mathbf{O}$ (and $C-UR$), $\Pi x(Rx \to \mathcal{W}_x\mathbb{G}(\Box A \to \mathbf{O}A))$ is valid in this model; and if it satisfies $C-\mathcal{W}\mathbb{G}\mathbf{O}\Diamond$ (and $C-UR$), $\Pi x(Rx \to \mathcal{W}_x\mathbb{G}(\mathbf{O}A \to \Diamond A))$ is valid in this model.

If a model satisfies $C-\mathbf{O}\mathbb{G}\Box\mathcal{W}$ and $C-\mathbf{O}\Box\mathcal{W}$ (and $C-UR$), $\Pi x(Rx \to \mathbf{O}\underline{\mathbb{G}}(\Box A \to \mathcal{W}_xA))$ is valid in this model; if it satisfies $C-\mathbf{O}\mathbb{G}\mathcal{W}\Diamond$ and $C-\mathbf{O}\mathcal{W}\Diamond$ (and $C-UR$), $\Pi x(Rx \to \mathbf{O}\underline{\mathbb{G}}(\mathcal{W}_xA \to \Diamond A))$ is valid in this model; if it satisfies $C-\mathcal{W}\mathbb{G}\Box\mathbf{O}$ and $C-\mathcal{W}\Box\mathbf{O}$ (and $C-UR$), $\Pi x(Rx \to \mathcal{W}_x\underline{\mathbb{G}}(\Box A \to \mathbf{O}A))$ is valid in this model; and if it satisfies $C-\mathcal{W}\mathbb{G}\mathbf{O}\Diamond$ and $C-\mathcal{W}\mathbf{O}\Diamond$ (and $C-UR$), $\Pi x(Rx \to \mathcal{W}_x\underline{\mathbb{G}}(\mathbf{O}A \to \Diamond A))$ is valid in this model.

We have now considered some possible interactions between the different accessibility relations in our models. It is also possible to impose conditions that involve the valuation function v. Let us consider four conditions of this kind.

The conditions in **Table 16** are concerned with some possible relations between the alethic accessibility relation $\mathfrak{R}$ and the valuation function v. 'FT' stands for 'forward transfer', 'BT' for 'backward transfer', 'R' for 'rationality' and 'U' for 'universal'. In every model that satisfies $C-UR$, we can show that every perfectly rational individual (at every moment in time) is necessarily perfectly rational; in every model that satisfies $C-FTR$ (and $C-\Box\mathcal{W}$), we can show that every perfectly rational individual (at every moment in time) wants to be perfectly rational. According to $C-FT$, every atomic formula (and every sentence of the form $\mathcal{W}_cB$, $\mathcal{A}_cB$, $\mathcal{R}_cB$, $\mathcal{I}_cB$ or $\mathcal{N}_cB$, given that $\neg Rc$ is true) is historically settled, and according to $C-BT$ every atomic formula (and every sentence of the form $\mathcal{W}_cB$, $\mathcal{A}_cB$, $\mathcal{R}_cB$, $\mathcal{I}_cB$ or $\mathcal{N}_cB$, given that $\neg Rc$ is true) that is historically possible is true. $C-FT$ and $C-BT$ are plausible if we think of the world as a tree-like structure. Note that they do not entail that *every* sentence is historically necessary, nor that *every* sentence that is historically possible is true. Even if we assume that $C-FT$ and $C-BT$ hold, various 'future-directed sentences', such as $\mathbb{G}Fa$ and $\mathbb{F}Ra$, are, for example, not necessarily historically settled.

The conditions mentioned in this section can be used to obtain a categorisation of the set of all models into various kinds. Let $\mathcal{M}(C_1, \ldots, C_n)$ be the class of (all) models that satisfy the conditions $C_1, \ldots, C_n$. Then, for example, $\mathcal{M}(C-bD, C-b4, C-b5)$ is the class of (all) models that satisfy the conditions $C-bD$, $C-b4$ and $C-b5$, etc.

We can now define the concept of a system of a class of models.

Definition 4 ***(System of a class of models)*** *The set of all sentences in the language $\mathcal{L}$ that are valid in a class of models $\mathcal{M}$ is called the (logical) system of $\mathcal{M}$, or the logic of $\mathcal{M}$, $\mathcal{S}(\mathcal{M})$.*

By imposing different conditions on our models we can generate many logical systems that are non-equivalent. $\mathcal{S}(\mathcal{M}(C-bD, C-b4, C-b5))$ (the system of $\mathcal{M}(C-bD, C-b4, C-b5)$) is, for example, the class of sentences in $\mathcal{L}$ that are valid in the class of (all) models that satisfy the conditions $C-bD$, $C-b4$ and $C-b5$.

We have now described the semantics of our systems. Let us turn to the proof theory.

4 Proof theory

In Section 4, I will describe several tableau rules that can be used to construct a set of tableau systems. Every tableau system is an extension of propositional logic (for more on semantic tableau and propositional logic, see, for example, [136] and [86]). Every system also includes a modal part, a temporal part, a deontic part, a boulesic part and rules for a pair of (possibilist) quantifiers. For more information on the tableau method and various kinds of tableau systems, see, for example, [48], [59] and [118].

The tableau rules in this section correspond to the semantic conditions introduced in Section 3.3. The interpretation of the rules is standard. For example, according to $\mathbb{U}$ (**Table 18**), we may add $A, w_j t_l$ (for any w_j and t_l) to any open branch in a tree that includes $\mathbb{U}A, w_i t_k$; according to $\neg\wedge$, we may extend the tip of any open branch in a tree on which $\neg(A \wedge B), w_i t_k$ occurs into two new branches, with $\neg A, w_i t_k$ at the tip of one new branch and $\neg B, w_i t_k$ at the other, etc.

Intuitively, '$Rc, w_i t_k$' in the 'boulesic rules' says that the individual denoted by 'c' is perfectly rational in the world denoted by 'w_i' at the time denoted by 't_k', and '$Acw_i w_j t_k$' says that the world denoted by 'w_j' is boulesically accessible (acceptable) to the individual denoted by 'c' in the world denoted by 'w_i' at the time denoted by 't_k'. Note that c can be replaced by any constant in the rules in **Table 21**–**22**. The same is true of other rules in this section that include something of the form $Acw_i w_j t_k$; that is, in every rule of this kind, c can be replaced by any constant.

The quantifier rules (**Table 23**) are never instantiated with variables; a in $A[a/x]$ is any constant on the branch (or a new one if there are no constants on the branch) and c in $A[c/x]$ is a constant that is new to the branch, that is, that does not already occur on the branch.

In the CUT rule (**Table 24**), A can be replaced by any sentence. However, in the completeness proofs, I will use a weaker rule, $CUTR$, and not CUT. In $CUTR$, A is of the form Rc, where c is a constant (that occurs as an index to some boulesic operator) on the branch. $T-Ii$ and $T-Iii$ are redundant in any system that does not include $T-FC$, $T-PC$ or $T-C$ (see **Table 27**).

$(T-TIi)$ is interpreted in the following way. $A(t_i)$ is a line in a tableau that includes 't_i', and $A(t_j)$ is like $A(t_i)$ except that 't_i' is replaced by 't_j'. That is, if $A(t_i)$ is $A, w_k t_i$,

Table 17: Propositional rules

$\neg\neg$	$\wedge$	$\neg\wedge$
$\neg\neg A, w_i t_k$ $\downarrow$ $A, w_i t_k$	$(A \wedge B), w_i t_k$ $\downarrow$ $A, w_i t_k$ $B, w_i t_k$	$\neg(A \wedge B), w_i t_k$ $\swarrow \searrow$ $\neg A, w_i t_k \quad \neg B, w_i t_k$
$\vee$	$\neg\vee$	$\rightarrow$
$(A \vee B), w_i t_k$ $\swarrow \searrow$ $A, w_i t_k \quad B, w_i t_k$	$\neg(A \vee B), w_i t_k$ $\downarrow$ $\neg A, w_i t_k$ $\neg B, w_i t_k$	$(A \rightarrow B), w_i t_k$ $\swarrow \searrow$ $\neg A, w_i t_k \quad B, w_i t_k$
$\neg \rightarrow$	$\leftrightarrow$	$\neg \leftrightarrow$
$\neg(A \rightarrow B), w_i t_k$ $\downarrow$ $A, w_i t_k$ $\neg B, w_i t_k$	$(A \leftrightarrow B), w_i t_k$ $\swarrow \searrow$ $A, w_i t_k \quad \neg A, w_i t_k$ $B, w_i t_k \quad \neg B, w_i t_k$	$\neg(A \leftrightarrow B), w_i t_k$ $\swarrow \searrow$ $A, w_i t_k \quad \neg A, w_i t_k$ $\neg B, w_i t_k \quad B, w_i t_k$

Table 18: Basic alethic rules (ba-rules)

$\mathbb{U}$	$\mathbb{M}$	$\Box$	$\Diamond$
$\mathbb{U}A, w_i t_k$ $\downarrow$ $A, w_j t_l$ for any w_j and t_l	$\mathbb{M}A, w_i t_k$ $\downarrow$ $A, w_j t_l$ where w_j and t_l are new	$\Box A, w_i t_k$ $r w_i w_j t_k$ $\downarrow$ $A, w_j t_k$	$\Diamond A, w_i t_k$ $\downarrow$ $r w_i w_j t_k$ $A, w_j t_k$ where w_j is new
$\neg\mathbb{U}$	$\neg\mathbb{M}$	$\neg\Box$	$\neg\Diamond$
$\neg\mathbb{U}A, w_i t_k$ $\downarrow$ $\mathbb{M}\neg A, w_i t_k$	$\neg\mathbb{M}A, w_i t_k$ $\downarrow$ $\mathbb{U}\neg A, w_i t_k$	$\neg \Box A, w_i t_k$ $\downarrow$ $\Diamond\neg A, w_i t_k$	$\neg \Diamond A, w_i t_k$ $\downarrow$ $\Box\neg A, w_i t_k$

Table 19: Basic temporal rules (bt-rules)

$\mathbb{A}$	$\neg\mathbb{A}$	$\mathbb{S}$	$\neg\mathbb{S}$
$\mathbb{A}A, w_it_j$ $\downarrow$ A, w_it_k for every t_k on the branch	$\neg\mathbb{A}A, w_it_j$ $\downarrow$ $\mathbb{S}\neg A, w_it_j$	$\mathbb{S}A, w_it_j$ $\downarrow$ A, w_it_k where t_k is new to the branch	$\neg\mathbb{S}A, w_it_j$ $\downarrow$ $\mathbb{A}\neg A, w_it_j$
$\mathbb{G}$	$\neg\mathbb{G}$	$\mathbb{F}$	$\neg\mathbb{F}$
$\mathbb{G}A, w_it_j$ $t_j < t_k$ $\downarrow$ A, w_it_k	$\neg\mathbb{G}A, w_it_j$ $\downarrow$ $\mathbb{F}\neg A, w_it_j$	$\mathbb{F}A, w_it_j$ $\downarrow$ $t_j < t_k$ A, w_it_k where t_k is new	$\neg\mathbb{F}A, w_it_j$ $\downarrow$ $\mathbb{G}\neg A, w_it_j$
$\mathbb{H}$	$\neg\mathbb{H}$	$\mathbb{P}$	$\neg\mathbb{P}$
$\mathbb{H}A, w_it_j$ $t_k < t_j$ $\downarrow$ A, w_it_k	$\neg\mathbb{H}A, w_it_j$ $\downarrow$ $\mathbb{P}\neg A, w_it_j$	$\mathbb{P}A, w_it_j$ $\downarrow$ $t_k < t_j$ A, w_it_k where t_k is new	$\neg\mathbb{P}A, w_it_j$ $\downarrow$ $\mathbb{H}\neg A, w_it_j$

Table 20: Basic deontic rules (bd-rules)

$\mathbf{O}$	$\mathbf{P}$	$\neg\mathbf{O}$	$\neg\mathbf{P}$
$\mathbf{O}B, w_it_k$ $sw_iw_jt_k$ $\downarrow$ B, w_jt_k	$\mathbf{P}B, w_it_k$ $\downarrow$ $sw_iw_jt_k$ B, w_jt_k where w_j is new	$\neg\mathbf{O}B, w_it_k$ $\downarrow$ $\mathbf{P}\neg B, w_it_k$	$\neg\mathbf{P}B, w_it_k$ $\downarrow$ $\mathbf{O}\neg B, w_it_k$

Table 21: Basic boulesic rules (bb-rules)

$\mathcal{W}$	$\mathcal{A}$	$\mathcal{R}$	$\neg\mathcal{W}$	$\neg\mathcal{A}$	$\neg\mathcal{R}$
Rc, w_it_k $\mathcal{W}_cB, w_it_k$ $Acw_iw_jt_k$ $\downarrow$ B, w_jt_k	Rc, w_it_k $\mathcal{A}_cB, w_it_k$ $\downarrow$ $Acw_iw_jt_k$ B, w_jt_k where w_j is new	Rc, w_it_k $\mathcal{R}_cB, w_it_k$ $Acw_iw_jt_k$ $\downarrow$ $\neg B, w_jt_k$	Rc, w_it_k $\neg\mathcal{W}_cB, w_it_k$ $\downarrow$ $\mathcal{A}_c\neg B, w_it_k$	Rc, w_it_k $\neg\mathcal{A}_cB, w_it_k$ $\downarrow$ $\mathcal{W}_c\neg B, w_it_k$	Rc, w_it_k $\neg\mathcal{R}_cB, w_it_k$ $\downarrow$ $\mathcal{A}_cB, w_it_k$

Table 22: Basic boulesic rules II (bb-rules)

$\mathcal{I}$	$\mathcal{N}$	$\neg\mathcal{I}$	$\neg\mathcal{N}$
$Rc, w_i t_k$ $\mathcal{I}_c B, w_i t_k$ $\downarrow$ $\mathcal{A}_c B, w_i t_k$ $\mathcal{A}_c \neg B, w_i t_k$	$Rc, w_i t_k$ $\mathcal{N}_c B, w_i t_k$ $\swarrow \searrow$ $\mathcal{W}_c B, w_i t_k \quad \mathcal{W}_c \neg B, w_i t_k$	$Rc, w_i t_k$ $\neg\mathcal{I}_c B, w_i t_k$ $\downarrow$ $\mathcal{N}_c B, w_i t_k$	$Rc, w_i t_k$ $\neg\mathcal{N}_c B, w_i t_k$ $\downarrow$ $\mathcal{I}_c B, w_i t_k$

Table 23: Possibilist quantifiers

Π	Σ	$\neg\Pi$	$\neg\Sigma$
$\Pi x A, w_i t_j$ $\downarrow$ $A[a/x], w_i t_j$ for every constant a on the branch, a new if there are no constants on the branch	$\Sigma x A, w_i t_j$ $\downarrow$ $A[c/x], w_i t_j$ where c is new to the branch	$\neg\Pi x A, w_i t_j$ $\downarrow$ $\Sigma x \neg A, w_i t_j$	$\neg\Sigma x A, w_i t_j$ $\downarrow$ $\Pi x \neg A, w_i t_j$

Table 24: The CUT-rule (CUT), ($CUTR$) and temporal identity rules

CUT	$T-TIi$	$T-TIii$
$w_i t_k$ $\swarrow \searrow$ $A, w_i t_k \quad \neg A, w_i t_k$ for every A, w_i and t_k	$A(t_i)$ $t_i = t_j$ $\downarrow$ $A(t_j)$	$A(t_i)$ $t_j = t_i$ $\downarrow$ $A(t_j)$

Table 25: Identity rules

$T-R=$	$T-S=$	$T-N=,$	$T-A=$
$*$	$s=t, w_it_k$	$a=b, w_it_k$	$a=b, w_it_l$
$\downarrow$	$A[s/x], w_it_k$	$\downarrow$	$Aaw_jw_kt_m$
$t=t, w_it_k$	$\downarrow$	$a=b, w_jt_l$	$\downarrow$
for every t	$A[t/x], w_it_k$	for any w_j and t_l	$Abw_jw_kt_m$
on the branch	where A		
	is of a certain form		
	(see explanation in the text)		

Table 26: Alethic accessibility rules (a-rules)

$T-aD$	$T-aT$	$T-aB$	$T-a4$	$T-a5$
w_it_k	w_it_k	$rw_iw_jt_k$	$rw_iw_jt_l$	$rw_iw_jt_l$
$\downarrow$	$\downarrow$	$\downarrow$	$rw_jw_kt_l$	$rw_iw_kt_l$
$rw_iw_jt_k$	$rw_iw_it_k$	$rw_jw_it_k$	$\downarrow$	$\downarrow$
where w_j is new			$rw_iw_kt_l$	$rw_jw_kt_l$

then $A(t_j)$ is A, w_kt_j; if $A(t_i)$ is $rw_kw_lt_i$, then $A(t_j)$ is $rw_kw_lt_j$; if $A(t_i)$ is $t_i = t_k$, then $A(t_j)$ is $t_j = t_k$, etc. If $A(t_i)$ is A, w_kt_i, we only apply the rule when A is atomic or of the form $\mathcal{W}_tD$, $\mathcal{A}_tD$, $\mathcal{R}_tD$, $\mathcal{I}_tD$ or $\mathcal{N}_tD$ given that $\neg Rt, w_kt_i$ is on the branch. $T-TIii$ is interpreted similarly.

Table 25 includes some 'identity rules'. R in ($T-R=$) stands for 'reflexive', S in ($T-S=$) for 'substitution (of identities)', N in ($T-N=$) for 'necessary identity', and A in ($T-A=$) for '(boulesic) accessibility'. ($T-R=$) is a rule without premises; $t=t, w_it_k$ may be added to any open branch in a tree.

($T-S=$) is applied only 'within world-moment pairs' and it may be applied to any atomic formula. However, we shall also allow substitutions of the following kind. Let M be a matrix where x_m is the first free variable in M and a_m is the constant in $M[a_1,\ldots,a,\ldots,a_n/\vec{x}]$ that replaces x_m. Furthermore, assume that we have $a=b, w_it_k$, $M[a_1,\ldots,a,\ldots,a_n/\vec{x}], w_it_k$ and $\neg Ra_m, w_it_k$ on the branch. Then we may apply ($T-S=$) to generate an extension of the branch that includes $M[a_1,\ldots,b,\ldots,a_n/\vec{x}], w_it_k$.

Table 42 includes some 'transfer rules'. 'FT' in '$T-FT$' and '$T-FTR$' is an abbreviation of 'Forward Transfer', 'BT' in '$T-BT$' of 'Backward Transfer', and 'R' in '$T-FTR$' and '$T-UR$' of 'Rationality'. Note that A in $T-FT$ ($T-BT$) is atomic or of the form $\mathcal{W}_cB$, $\mathcal{A}_cB$, $\mathcal{R}_cB$, $\mathcal{I}_cB$ or $\mathcal{N}_cB$ given that $\neg Rc, w_it_l$ ($\neg Rc, w_jt_l$) is on the branch. $T-FT$ is stronger than $T-FTR$; $T-FTR$ is derivable in every system that includes $T-FT$.

If a system includes $T-UR$, we can show that the following sentence is a theorem in this system: $\Pi x(Rx \to \mathbb{U}Rx)$, which says that every perfectly rational individual is necessarily perfectly rational. Intuitively, it is not obvious that this principle is true. Individuals that are contingently perfectly rational are conceivable and appear to be

Table 27: Temporal accessibility rules (t-rules)

$T-t4$	$T-PD$	$T-FD$
$t_i < t_j$ $t_j < t_k$ $\downarrow$ $t_i < t_k$	t_j $\downarrow$ $t_k < t_j$ where t_k is new	t_j $\downarrow$ $t_j < t_k$ where t_k is new
$T-DE$	$T-FC$	$T-PC$
$t_i < t_j$ $\downarrow$ $t_i < t_k$ $t_k < t_j$ where t_k is new	$t_i < t_j$ $t_i < t_k$ $\swarrow\downarrow\searrow$ $t_j < t_k$ $t_j = t_k$ $t_k < t_j$	$t_j < t_i$ $t_k < t_i$ $\swarrow\downarrow\searrow$ $t_j < t_k$ $t_j = t_k$ $t_k < t_j$
$T-C$	$T-UB$	$T-LB$
t_i, t_j $\swarrow \quad \downarrow \quad \searrow$ $t_i < t_j$ $\quad t_i = t_j$ $\quad t_j < t_i$	$t_i < t_j$ $t_i < t_k$ $\downarrow$ $t_j < t_l$ $t_k < t_l$ where t_l is new to the branch	$t_j < t_i$ $t_k < t_i$ $\downarrow$ $t_l < t_j$ $t_l < t_k$ where t_l is new to the branch

Table 28: Deontic accessibility rules (d-rules)

$T-dD$	$T-d4$	$T-d5$	$T-\mathbf{O}dT$	$T-\mathbf{O}dB$
w_it_k $\downarrow$ $sw_iw_jt_k$ where w_j is new	$sw_iw_jt_l$ $sw_jw_kt_l$ $\downarrow$ $sw_iw_kt_l$	$sw_iw_jt_l$ $sw_iw_kt_l$ $\downarrow$ $sw_jw_kt_l$	$sw_iw_jt_l$ $\downarrow$ $sw_jw_jt_l$	$sw_iw_jt_l$ $sw_jw_kt_l$ $\downarrow$ $sw_kw_jt_l$

Table 29: Boulesic accessibility rules (b-rules)

$T-bD$	$T-b4$	$T-b5$	$T-\mathcal{W}bT$	$T-\mathcal{W}bB$
w_it_k $\downarrow$ $Acw_iw_jt_k$ where w_j is new	$Acw_iw_jt_l$ $Acw_jw_kt_l$ $\downarrow$ $Acw_iw_kt_l$	$Acw_iw_jt_l$ $Acw_iw_kt_l$ $\downarrow$ $Acw_jw_kt_l$	$Acw_iw_jt_l$ $\downarrow$ $Acw_jw_jt_l$	$Acw_iw_jt_l$ $Acw_jw_kt_l$ $\downarrow$ $Acw_kw_jt_l$

Table 30: Alethic-deontic accessibility rules (ad-rules)

$T-\Box\mathbf{O}$	$T-\mathbf{O}\Diamond$	$T-\mathbf{O}\Box\mathbf{O}$	$T-\mathbf{O}\mathbf{O}\Diamond$	$T-da4$	$T-da5$
$sw_iw_jt_k$ $\downarrow$ $rw_iw_jt_k$	w_it_k $\downarrow$ $sw_iw_jt_k$ $rw_iw_jt_k$ where w_j is new	$sw_iw_jt_l$ $sw_jw_kt_l$ $\downarrow$ $rw_jw_kt_l$	$sw_iw_jt_l$ $\downarrow$ $rw_jw_kt_l$ $sw_jw_kt_l$ where w_k is new	$sw_iw_jt_l$ $rw_jw_kt_l$ $\downarrow$ $rw_iw_kt_l$	$sw_iw_jt_l$ $rw_iw_kt_l$ $\downarrow$ $rw_jw_kt_l$
$T-ad4$	$T-ad5$	$T-\mathbf{P}\Box P$	$T-\mathbf{O}\Box P$	$T-\Box\mathbf{O}P$	
$rw_iw_jt_l$ $sw_jw_kt_l$ $\downarrow$ $sw_iw_kt_l$	$rw_iw_jt_l$ $sw_iw_kt_l$ $\downarrow$ $sw_jw_kt_l$	$sw_iw_jt_m$ $rw_iw_kt_m$ $\downarrow$ $rw_jw_lt_m$ $sw_kw_lt_m$ where w_l is new	$rw_iw_jt_m$ $sw_jw_kt_m$ $\downarrow$ $sw_iw_lt_m$ $rw_lw_kt_m$ where w_l is new	$sw_iw_jt_m$ $rw_jw_kt_m$ $\downarrow$ $rw_iw_lt_m$ $sw_lw_kt_m$ where w_l is new	

Table 31: Alethic-boulesic accessibility rules (ab-rules)

$T-\Box\mathcal{W}$	$T-\mathcal{W}\Diamond$	$T-\mathcal{W}\Box\mathcal{W}$	$T-\mathcal{W}\mathcal{W}\Diamond$	$T-ba4$	$T-ba5$
$Acw_iw_jt_k$ $\downarrow$ $rw_iw_jt_k$	w_it_k $\downarrow$ $Acw_iw_jt_k$ $rw_iw_jt_k$ where w_j is new	$Acw_iw_jt_l$ $Acw_jw_kt_l$ $\downarrow$ $rw_jw_kt_l$	$Acw_iw_jt_l$ $\downarrow$ $rw_jw_kt_l$ $Acw_jw_kt_l$ where w_k is new	$Acw_iw_jt_l$ $rw_jw_kt_l$ $\downarrow$ $rw_iw_kt_l$	$Acw_iw_jt_l$ $rw_iw_kt_l$ $\downarrow$ $rw_jw_kt_l$
$T-ab4$	$T-ab5$	$T-\mathcal{A}\Box P$	$T-\mathcal{W}\Box P$	$T-\Box\mathcal{W}P$	
$rw_iw_jt_l$ $Acw_jw_kt_l$ $\downarrow$ $Acw_iw_kt_l$	$rw_iw_jt_l$ $Acw_iw_kt_l$ $\downarrow$ $Acw_jw_kt_l$	$Acw_iw_jt_m$ $rw_iw_kt_m$ $\downarrow$ $rw_jw_lt_m$ $Acw_kw_lt_m$ where w_l is new	$rw_iw_jt_m$ $Acw_jw_kt_m$ $\downarrow$ $Acw_iw_lt_m$ $rw_lw_kt_m$ where w_l is new	$Acw_iw_jt_m$ $rw_jw_kt_m$ $\downarrow$ $rw_iw_lt_m$ $Acw_lw_kt_m$ where w_l is new	

Table 32: Boulesic-deontic accessibility rules (b-d-rules)

$T-\mathbf{O}\mathcal{W}$	$T-\mathcal{W}\mathbf{O}$	$T-\mathbf{O}\mathcal{A}$	$T-bd4$	$T-bd5$
$Acw_iw_jt_k$ $\downarrow$ $sw_iw_jt_k$	$sw_iw_jt_k$ $\downarrow$ $Acw_iw_jt_k$	w_it_k $\downarrow$ $sw_iw_jt_k$ $Acw_iw_jt_k$	$sw_iw_jt_l$ $Acw_jw_kt_l$ $\downarrow$ $Acw_iw_kt_l$	$sw_iw_jt_l$ $Acw_iw_kt_l$ $\downarrow$ $Acw_jw_kt_l$
$T-db4$	$T-db5$	$T-\mathcal{A}\mathbf{O}P$	$T-\mathcal{W}\mathbf{O}P$	$T-\mathbf{O}\mathcal{W}P$
$Acw_iw_jt_l$ $sw_jw_kt_l$ $\downarrow$ $sw_iw_kt_l$	$Acw_iw_jt_l$ $sw_iw_kt_l$ $\downarrow$ $sw_jw_kt_l$	$Acw_iw_jt_m$ $sw_iw_kt_m$ $\downarrow$ $sw_jw_lt_m$ $Acw_kw_lt_m$ where w_l is new	$sw_iw_jt_m$ $Acw_jw_kt_m$ $\downarrow$ $Acw_iw_lt_m$ $sw_lw_kt_m$ where w_l is new	$Acw_iw_jt_m$ $sw_jw_kt_m$ $\downarrow$ $sw_iw_lt_m$ $Acw_lw_kt_m$ where w_l is new

Table 33: Boulesic-deontic accessibility rules (b-d-rules)

$T-\mathbf{O}\mathbf{O}\mathcal{W}$	$T-\mathbf{O}\mathcal{W}\mathbf{O}$	$T-\mathbf{O}\mathbf{O}\mathcal{A}$
$sw_iw_jt_l$ $Acw_jw_kt_l$ $\downarrow$ $sw_jw_kt_l$	$sw_iw_jt_l$ $sw_jw_kt_l$ $\downarrow$ $Acw_jw_kt_l$	$sw_iw_jt_l$ $\downarrow$ $Acw_jw_kt_l$ $sw_jw_kt_l$ where w_k is new
$T-\mathcal{W}\mathbf{O}\mathcal{W}$	$T-\mathcal{W}\mathcal{W}\mathbf{O}$	$T-\mathcal{W}\mathbf{O}\mathcal{A}$
$Acw_iw_jt_l$ $Acw_jw_kt_l$ $\downarrow$ $sw_jw_kt_l$	$Acw_iw_jt_l$ $sw_jw_kt_l$ $\downarrow$ $Acw_jw_kt_l$	$Acw_iw_jt_l$ $\downarrow$ $Acw_jw_kt_l$ $sw_jw_kt_l$ where w_k is new

Table 34: Temporal-alethic accessibility rules

$T-ASP$	$T-AR$
$rw_iw_jt_l$ $t_k < t_l$ $\downarrow$ $rw_iw_jt_k$	$rw_iw_jt_l$ $t_l < t_m$ $rw_jw_kt_m$ $\downarrow$ $rw_iw_kt_l$

Table 35: Temporal-deontic accessibility rules

$T-\mathbf{O}\mathbb{G}dT$	$T-\mathbf{O}\mathbb{G}dB$	$T-DR$
$sw_iw_jt_l$	$sw_iw_jt_l$	$sw_iw_jt_l$
$t_l < t_m$	$t_l < t_m$	$t_l < t_m$
$\downarrow$	$sw_jw_kt_m$	$sw_jw_kt_m$
$sw_jw_jt_m$	$\downarrow$	$\downarrow$
	$sw_kw_jt_m$	$sw_iw_kt_l$

Table 36: Temporal-boulesic accessibility rules

$T-\mathcal{W}\mathbb{G}bT$	$T-\mathcal{W}\mathbb{G}bB$	$T-BR$
$Acw_iw_jt_l$	$Acw_iw_jt_l$	$Acw_iw_jt_l$
$t_l < t_m$	$t_l < t_m$	$t_l < t_m$
$\downarrow$	$Acw_jw_kt_m$	$Acw_jw_kt_m$
$Acw_jw_jt_m$	$\downarrow$	$\downarrow$
	$Acw_kw_jt_m$	$Acw_iw_kt_l$

Table 37: Alethic-boulesic-deontic accessibility rules

$T-\mathbf{O}\Box\mathcal{W}$	$T-\mathcal{W}\Box\mathbf{O}$	$T-\mathbf{O}\mathcal{W}\Diamond$	$T-\mathcal{W}\mathbf{O}\Diamond$
$sw_iw_jt_l$	$Acw_iw_jt_l$	$sw_iw_jt_l$	$Acw_iw_jt_l$
$Acw_jw_kt_l$	$sw_jw_kt_l$	$\downarrow$	$\downarrow$
$\downarrow$	$\downarrow$	$Acw_jw_kt_l$	$sw_jw_kt_l$
$rw_jw_kt_l$	$rw_jw_kt_l$	$rw_jw_kt_l$	$rw_jw_kt_l$
		where w_k is new	where w_k is new

Table 38: Temporal-boulesic-deontic accessibility rules

$T-\mathbf{O}\mathbb{G}bT$	$T-\mathbf{O}\mathbb{G}bB$	$T-\mathbf{O}\mathbb{G}\mathbf{O}\mathcal{W}$	$T-\mathbf{O}\mathbb{G}\mathcal{W}\mathbf{O}$	$T-\mathbf{O}\mathbb{G}\mathbf{O}\mathcal{A}$
$sw_iw_jt_l$	$sw_iw_jt_l$	$sw_iw_jt_l$	$sw_iw_jt_l$	$sw_iw_jt_l$
$t_l < t_m$	$t_l < t_m$	$t_l < t_m$	$t_l < t_m$	$t_l < t_m$
$\downarrow$	$Acw_jw_kt_m$	$Acw_jw_kt_m$	$sw_jw_kt_m$	$\downarrow$
$Acw_jw_jt_m$	$\downarrow$	$\downarrow$	$\downarrow$	$Acw_jw_kt_m$
	$Acw_kw_jt_m$	$sw_jw_kt_m$	$Acw_jw_kt_m$	$sw_jw_kt_m$
				where w_k is new
$T-\mathcal{W}\mathbb{G}dT$	$T-\mathcal{W}\mathbb{G}dB$	$T-\mathcal{W}\mathbb{G}\mathbf{O}\mathcal{W}$	$T-\mathcal{W}\mathbb{G}\mathcal{W}\mathbf{O}$	$T-\mathcal{W}\mathbb{G}\mathbf{O}\mathcal{A}$
$Acw_iw_jt_l$	$Acw_iw_jt_l$	$Acw_iw_jt_l$	$Acw_iw_jt_l$	$Acw_iw_jt_l$
$t_l < t_m$	$t_l < t_m$	$t_l < t_m$	$t_l < t_m$	$t_l < t_m$
$\downarrow$	$sw_jw_kt_m$	$Acw_jw_kt_m$	$sw_jw_kt_m$	$\downarrow$
$sw_jw_jt_m$	$\downarrow$	$\downarrow$	$\downarrow$	$Acw_jw_kt_m$
	$sw_kw_jt_m$	$sw_jw_kt_m$	$Acw_jw_kt_m$	$sw_jw_kt_m$
				where w_k is new

Table 39: Temporal-alethic-deontic accessibility rules

$T - \mathbf{O}\mathbb{G}\Box\mathbf{O}$	$T - \mathbf{O}\mathbb{G}\mathbf{O}\Diamond$
$sw_iw_jt_l$	$sw_iw_jt_l$
$t_l < t_m$	$t_l < t_m$
$sw_jw_kt_m$	$\downarrow$
$\downarrow$	$rw_jw_kt_m$
$rw_jw_kt_m$	$sw_jw_kt_m$
	where w_k
	is new

Table 40: Temporal-alethic-boulesic accessibility rules

$T - \mathcal{W}\mathbb{G}\Box\mathcal{W}$	$T - \mathcal{W}\mathbb{G}\mathcal{W}\Diamond$
$Acw_iw_jt_l$	$Acw_iw_jt_l$
$t_l < t_m$	$t_l < t_m$
$Acw_jw_kt_m$	$\downarrow$
$\downarrow$	$rw_jw_kt_m$
$rw_jw_kt_m$	$Acw_jw_kt_m$
	where w_k
	is new

Table 41: Temporal-alethic-boulesic-deontic accessibility rules

$T - \mathbf{O}\mathbb{G}\Box\mathcal{W}$	$T - \mathbf{O}\mathbb{G}\mathcal{W}\Diamond$	$T - \mathcal{W}\mathbb{G}\Box\mathbf{O}$	$T - \mathcal{W}\mathbb{G}\mathbf{O}\Diamond$
$sw_iw_jt_l$	$sw_iw_jt_l$	$Acw_iw_jt_l$	$Acw_iw_jt_l$
$t_l < t_m$	$t_l < t_m$	$t_l < t_m$	$t_l < t_m$
$Acw_jw_kt_m$	$\downarrow$	$sw_jw_kt_m$	$\downarrow$
$\downarrow$	$rw_jw_kt_m$	$\downarrow$	$rw_jw_kt_m$
$rw_jw_kt_m$	$Acw_jw_kt_m$	$rw_jw_kt_m$	$sw_jw_kt_m$
	where w_k		where w_k
	is new		is new

Table 42: Transfer rules, etc.

$T - FT$	$T - BT$	$T - FTR$	$T - UR$
A, w_it_l	A, w_jt_l	Rc, w_it_k	Rc, w_it_k
$rw_iw_jt_l$	$rw_iw_jt_l$	$rw_iw_jt_k$	$\downarrow$
$\downarrow$	$\downarrow$	$\downarrow$	Rc, w_jt_l
A, w_jt_l	A, w_it_l	Rc, w_jt_k	for any w_j and t_l
where A is of	where A is of		
a certain form	a certain form		
(see explanation in the text)	(see explanation in the text)		

(logically) possible. So, it is a good thing that not all systems assume that this principle is true. Whether the transfer rules in **Table 42** should be added to our systems appears to be a matter of choice. $T - FT$ and $T - BT$ seem to be plausible if we think of reality as a tree like structure that branches towards the future but is determined in the past (and the present). But for some applications, we may want to exclude them. (For more on this, see Section 5.)

Let us now introduce some important concepts.

A tree is a kind of structure that consists of a set of *nodes* ordered by a successor relation. Every tree has a *root* that is a node that is not a successor of any node. Every other node in a tree is a successor of the root. A node without successors is a *tip* or a *leaf*. A path from the root to a tip is called a *branch*. For more on the concept of a tree, see [135] and [136], pp. 3–4.

A (semantic) tableau is a tree where the nodes have the following shape: $A, w_i t_j$, where A is a formula in $\mathcal{L}$ and $i, j \in \{0, 1, 2, 3, \ldots\}$, or $rw_i w_j t_k$, $sw_i w_j t_k$, $Acw_i w_j t_k$, $t_i < t_j$, or $t_i = t_j$ where $i, j, k \in \{0, 1, 2, 3, \ldots\}$ and c is a constant in $\mathcal{L}$.

A branch in a tableau is closed just in case both $A, w_i t_j$ and $\neg A, w_i t_j$ occur on the branch (for some A, w_i and t_j); it is open iff it is not closed. Intuitively, this means that a branch is closed iff it contains a contradiction and it is open precisely when it does not contain any contradiction. A tableau is closed iff every branch in it is closed; it is open just in case it is not closed.

Semantic tableaux can be used to test whether or not a sentence or argument is valid. Intuitively, a tableau rule shows us how to 'extend branches' from given nodes in a way that preserves satisfiability. We can think of the construction of a tableau as a systematic search for a model that makes the sentence we are testing false or the argument that we are testing invalid. If the tableau is closed, it is not possible to find a model of this kind, since it is not possible to find a consistent model in which all sentences on some branch in the tableau are true. Hence, if it is a sentence that we are testing, this sentence cannot be false, and if it is an argument that we are testing, this argument cannot have true premises and a false conclusion. Consequently, the sentence or argument that we are investigating is valid. If a branch in a tree is open (and every rule that can be applied has been applied), it is possible to use this branch to read off a countermodel to the sentence or argument that we are interested in. This countermodel shows that it is possible that the sentence we are testing is false or that the argument that we are testing has true premises and a false conclusion. Accordingly, this sentence or argument is invalid.[8]

Let us now define some important proof-theoretical concepts.

Definition 5 ***(Tableau systems) Tableau system:*** *A tableau system is a class of tableau rules.* ***Quantified temporal alethic boulesic deontic tableau system:*** *a quantified temporal alethic boulesic deontic tableau system is a tableau system that includes all propositional rules, all basic temporal rules, all basic alethic rules, all basic boulesic rules, all basic deontic rules, the rules for the possibilist quantifiers, the CUT-rule (or $CUTR$) and all the identity rules.*

The smallest quantified temporal alethic boulesic deontic tableau system is called

[8]This 'intuitive' line of thought is developed in more detail in the section on soundness and completeness.

$\mathcal{Q}$. By adding various tableau rules, it is possible to construct a large class of stronger quantified temporal alethic boulesic deontic tableau systems. Let the name of a quantified temporal alethic boulesic deontic tableau system be a list of the names of the (non-basic) rules that the system contains. The initial 'T' in a tableau rule may usually be omitted. So, '$aTdDbD$', for example, is the name of the quantified temporal alethic boulesic deontic tableau system that includes all the rules that every system of this kind contains and the rules $T-aT$, $T-dD$ and $T-bD$, etc.

Definition 6 ***(Proof-theoretical concepts)*** *In the following definitions, let S be a (quantified temporal alethic boulesic deontic) tableau system and let an S-tableau be a tableau generated in accordance with the rules in S.* ***Proof in a system:*** *A proof of a sentence A in S is a closed S-tableau for $\neg A, w_0t_0$, that is, a closed S-tableau that starts with $\neg A, w_0t_0$.* ***Theorem in a system:*** *A sentence A is a theorem in S (is provable in S) iff there is a proof of A in S, that is, iff there is a closed S-tableau for $\neg A, w_0t_0$.* ***Derivation in a system:*** *A derivation of a sentence B from a set of sentences Γ in S is a closed S-tableau that starts with A, w_0t_0 for every $A \in \Gamma$ and $\neg B, w_0t_0$. The sentences in Γ are called the premises and B the conclusion of the derivation.* ***Proof-theoretic consequence in a system:*** *The sentence B is a proof-theoretic consequence of the set of sentences Γ in S (B is derivable from Γ in S, $\Gamma \vdash_S B$) iff there is a derivation of B from Γ in S, that is, just in case there is a closed S-tableau that starts with A, w_0t_0 for every $A \in \Gamma$ and $\neg B, w_0t_0$.*

Definition 7 ***(The logic of a tableau system)*** *The logic $L(S)$ of a tableau system S is the class of all sentences in $\mathcal{L}$ that are provable in this system.*

For example, $L(aTdDbD)$, the logic of $aTdDbD$, is the class of all sentences in $\mathcal{L}$ that are provable in $aTdDbD$, that is, in the quantified temporal alethic boulesic deontic tableau system that includes all the rules that every system of this kind contains and the rules $T-aT$, $T-dD$ and $T-bD$.

5 Examples of theorems

In this section, I will mention some interesting formulas that are theorems in some tableau systems. The proofs are usually straightforward and are omitted.

Some 'boulesic' sentences that are theorems in every system. All the following sentences (schemas) are theorems in every system in this paper: $\Pi x(Rx \to (\mathcal{W}_xB \leftrightarrow \neg\mathcal{A}_x\neg B))$, $\Pi x(Rx \to (\neg\mathcal{W}_xB \leftrightarrow \mathcal{A}_x\neg B))$, $\Pi x(Rx \to (\mathcal{W}_x\neg B \leftrightarrow \neg\mathcal{A}_xB))$, $\Pi x(Rx \to (\mathcal{A}_xB \leftrightarrow \neg\mathcal{W}_x\neg B))$, $\Pi x(Rx \to (\mathcal{W}_x(A \wedge B) \leftrightarrow (\mathcal{W}_xA \wedge \mathcal{W}_xB)))$, $\Pi x(Rx \to ((\mathcal{W}_xA \vee \mathcal{W}_xB) \to \mathcal{W}_x(A \vee B)))$, $\Pi x(Rx \to (\mathcal{A}_x(A \wedge B) \to (\mathcal{A}_xA \wedge \mathcal{A}_xB)))$, $\Pi x(Rx \to (\mathcal{A}_x(A \vee B) \leftrightarrow (\mathcal{A}_xA \vee \mathcal{A}_xB)))$, $\Pi x(Rx \to (\mathcal{W}_x(A \to B) \to (\mathcal{W}_xA \to \mathcal{W}_xB)))$, $\Pi x(Rx \to ((\mathcal{W}_xA \wedge \mathcal{W}_x(A \to B)) \to \mathcal{W}_xB))$, $\Pi x(Rx \to (\mathcal{W}_x(A \to B) \to (\neg\mathcal{W}_xB \to \neg\mathcal{W}_xA)))$, $\Pi x(Rx \to ((\neg\mathcal{W}_xB \wedge \mathcal{W}_x(A \to B)) \to \neg\mathcal{W}_xA))$, $\Pi x(Rx \to (\mathcal{W}_x(A \to B) \to (\mathcal{W}_x\neg B \to \mathcal{W}_x\neg A)))$, $\Pi x(Rx \to ((\mathcal{W}_x\neg B \wedge \mathcal{W}_x(A \to B)) \to \mathcal{W}_x\neg A))$, $\Pi x(Rx \to (\mathcal{W}_x(A \to B) \to (\mathcal{A}_xA \to \mathcal{A}_xB)))$, $\Pi x(Rx \to ((\mathcal{A}_xA \wedge \mathcal{W}_x(A \to B)) \to \mathcal{A}_xB))$, $\Pi x(Rx \to (\mathcal{W}_x(A \to B) \to (\neg\mathcal{A}_xB \to \neg\mathcal{A}_xA)))$, $\Pi x(Rx \to ((\neg\mathcal{A}_xB \wedge \mathcal{W}_x(A \to B)) \to \neg\mathcal{A}_xA))$, $\Pi x(Rx \to (\mathcal{W}_x(A \to B) \to (\mathcal{A}_x\neg B \to \mathcal{A}_x\neg A)))$, $\Pi x(Rx \to ((\mathcal{A}_x\neg B \wedge \mathcal{W}_x(A \to B)) \to$

$\mathcal{A}_x\neg A))$, $\Pi x(Rx \to (\mathcal{W}_x(A \leftrightarrow B) \to (\mathcal{W}_xA \leftrightarrow \mathcal{W}_xB)))$, $\Pi x(Rx \to (\mathcal{W}_x(A \leftrightarrow B) \to (\neg\mathcal{W}_xA \leftrightarrow \neg\mathcal{W}_xB)))$, $\Pi x(Rx \to (\mathcal{W}_x(A \leftrightarrow B) \to (\mathcal{W}_x\neg A \leftrightarrow \mathcal{W}_x\neg B)))$, $\Pi x(Rx \to (\mathcal{W}_x(A \leftrightarrow B) \to (\mathcal{A}_xA \leftrightarrow \mathcal{A}_xB)))$, $\Pi x(Rx \to (\mathcal{W}_x(A \leftrightarrow B) \to (\neg\mathcal{A}_xA \leftrightarrow \neg\mathcal{A}_xB)))$, $\Pi x(Rx \to (\mathcal{W}_x(A \leftrightarrow B) \to (\mathcal{A}_x\neg A \leftrightarrow \mathcal{A}_x\neg B)))$, $\Pi x(Rx \to (\mathcal{R}_xB \leftrightarrow \neg\mathcal{A}_xB))$, $\Pi x(Rx \to (\mathcal{R}_xB \leftrightarrow \mathcal{W}_x\neg B))$, $\Pi x(Rx \to (\mathcal{N}_xB \leftrightarrow \neg\mathcal{I}_xB))$, $\Pi x(Rx \to (\mathcal{I}_xB \leftrightarrow (\mathcal{A}_xB \wedge \mathcal{A}_x\neg B)))$, $\Pi x(Rx \to (\mathcal{N}_xB \leftrightarrow (\mathcal{W}_xB \vee \mathcal{W}_x\neg B)))$, $\Pi x(Rx \to (\mathcal{I}_xB \leftrightarrow \mathcal{I}_x\neg B))$, $\Pi x(Rx \to (\mathcal{N}_xB \leftrightarrow \mathcal{N}_x\neg B))$, $\Pi x(Rx \to (\mathcal{W}_x(A \to B) \to (\mathcal{R}_xB \to \mathcal{R}_xA)))$, $\Pi x(Rx \to ((\mathcal{R}_xB \wedge \mathcal{W}_x(A \to B)) \to \mathcal{R}_xA))$, $\Pi x(Rx \to (\mathcal{W}_x(A \to B) \to (\neg\mathcal{R}_xA \to \neg\mathcal{R}_xB)))$, $\Pi x(Rx \to ((\neg\mathcal{R}_xA \wedge \mathcal{W}_x(A \to B)) \to \neg\mathcal{R}_xB))$, $\Pi x(Rx \to (\mathcal{W}_x(A \to B) \to (\mathcal{R}_x\neg A \to \mathcal{R}_x\neg B)))$, $\Pi x(Rx \to ((\mathcal{R}_x\neg A \wedge \mathcal{W}_x(A \to B)) \to \mathcal{R}_x\neg B))$, $\Pi x(Rx \to (\mathcal{W}_x(A \leftrightarrow B) \to (\mathcal{R}_xA \leftrightarrow \mathcal{R}_xB)))$, $\Pi x(Rx \to (\mathcal{W}_x(A \leftrightarrow B) \to (\neg\mathcal{R}_xA \leftrightarrow \neg\mathcal{R}_xB)))$, $\Pi x(Rx \to (\mathcal{W}_x(A \leftrightarrow B) \to (\mathcal{R}_x\neg A \leftrightarrow \mathcal{R}_x\neg B)))$, $\Pi x(Rx \to (\mathcal{W}_x(A \leftrightarrow B) \to (\mathcal{I}_xA \leftrightarrow \mathcal{I}_xB)))$, $\Pi x(Rx \to (\mathcal{W}_x(A \leftrightarrow B) \to (\neg\mathcal{I}_xA \leftrightarrow \neg\mathcal{I}_xB)))$, $\Pi x(Rx \to (\mathcal{W}_x(A \leftrightarrow B) \to (\mathcal{I}_x\neg A \leftrightarrow \mathcal{I}_x\neg B)))$, $\Pi x(Rx \to (\mathcal{W}_x(A \leftrightarrow B) \to (\mathcal{N}_xA \leftrightarrow \mathcal{N}_xB)))$, $\Pi x(Rx \to (\mathcal{W}_x(A \leftrightarrow B) \to (\neg\mathcal{N}_xA \leftrightarrow \neg\mathcal{N}_xB)))$, $\Pi x(Rx \to (\mathcal{W}_x(A \leftrightarrow B) \to (\mathcal{N}_x\neg A \leftrightarrow \mathcal{N}_x\neg B)))$.

Some sentences that include one type of operator that can be proved in every system. The dual of $\mathbb{U}$ is $\mathbb{M}$, the dual of $\Box$ is $\Diamond$, the dual of $\mathbb{A}$ is $\mathbb{S}$, the dual of $\mathbb{G}$ is $\mathbb{F}$, the dual of $\mathbb{H}$ is $\mathbb{P}$, the dual of **O** is **P**, the dual of $\underline{\mathbb{G}}$ is $\underline{\mathbb{F}}$, and the dual of $\underline{\mathbb{H}}$ is $\underline{\mathbb{P}}$. Let $\blacksquare$ be $\mathbb{U}$, $\Box$, $\mathbb{A}$, $\mathbb{G}$, $\mathbb{H}$, **O**, $\underline{\mathbb{G}}$ or $\underline{\mathbb{H}}$, and let $\blacklozenge$ be the dual of $\blacksquare$. Then, all the following sentences (schemas) are theorems in every system in this paper: $\blacksquare B \leftrightarrow \neg\blacklozenge\neg B$, $\neg\blacksquare B \leftrightarrow \blacklozenge\neg B$, $\blacksquare\neg B \leftrightarrow \neg\blacklozenge B$, $\blacklozenge B \leftrightarrow \neg\blacksquare\neg B$, $\blacksquare(A \wedge B) \leftrightarrow (\blacksquare A \wedge \blacksquare B)$, $(\blacksquare A \vee \blacksquare B) \to \blacksquare(A \vee B)$, $\blacklozenge(A \wedge B) \to (\blacklozenge A \wedge \blacklozenge B)$, $\blacklozenge(A \vee B) \leftrightarrow (\blacklozenge A \vee \blacklozenge B)$, $\blacksquare(A \to B) \to (\blacksquare A \to \blacksquare B)$, $(\blacksquare A \wedge \blacksquare(A \to B)) \to \blacksquare B$, $\blacksquare(A \to B) \to (\neg\blacksquare B \to \neg\blacksquare A)$, $(\neg\blacksquare B \wedge \blacksquare(A \to B)) \to \neg\blacksquare A$, $\blacksquare(A \to B) \to (\blacksquare\neg B \to \blacksquare\neg A)$, $(\blacksquare\neg B \wedge \blacksquare(A \to B)) \to \blacksquare\neg A$, $\blacksquare(A \to B) \to (\blacklozenge A \to \blacklozenge B)$, $(\blacklozenge A \wedge \blacksquare(A \to B)) \to \blacklozenge B$, $\blacksquare(A \to B) \to (\neg\blacklozenge B \to \neg\blacklozenge A)$, $(\neg\blacklozenge B \wedge \blacksquare(A \to B)) \to \neg\blacklozenge A$, $\blacksquare(A \to B) \to (\blacklozenge\neg B \to \blacklozenge\neg A)$, $(\blacklozenge\neg B \wedge \blacksquare(A \to B)) \to \blacklozenge\neg A$, $\blacksquare(A \leftrightarrow B) \to (\blacksquare A \leftrightarrow \blacksquare B)$, $\blacksquare(A \leftrightarrow B) \to (\neg\blacksquare A \leftrightarrow \neg\blacksquare B)$, $\blacksquare(A \leftrightarrow B) \to (\blacksquare\neg A \leftrightarrow \blacksquare\neg B)$, $\blacksquare(A \leftrightarrow B) \to (\blacklozenge A \leftrightarrow \blacklozenge B)$, $\blacksquare(A \leftrightarrow B) \to (\neg\blacklozenge A \leftrightarrow \neg\blacklozenge B)$, $\blacksquare(A \leftrightarrow B) \to (\blacklozenge\neg A \leftrightarrow \blacklozenge\neg B)$.

Some 'boulesic' sentences that are theorems in every *bD*-system. All the following sentences (schemas) are theorems in every system in this paper that includes *bD*: $\Pi x(Rx \to (\mathcal{W}_xB \to \mathcal{A}_xB))$, $\Pi x(Rx \to \neg(\mathcal{W}_xB \wedge \mathcal{W}_x\neg B))$, $\Pi x(Rx \to (\mathcal{A}_xB \vee \mathcal{A}_x\neg B))$, $\Pi x(Rx \to \neg(\mathcal{W}_x(A \vee B) \wedge (\mathcal{W}_x\neg A \wedge \mathcal{W}_x\neg B)))$, $\Pi x(Rx \to (\mathcal{W}_x(A \to B) \to (\mathcal{W}_xA \to \mathcal{A}_xB)))$, $\Pi x(Rx \to ((\mathcal{W}_xA \wedge \mathcal{W}_x(A \to B)) \to \mathcal{A}_xB))$, $\Pi x(Rx \to (\mathcal{W}_x(A \to B) \to (\neg\mathcal{A}_xB \to \neg\mathcal{W}_xA)))$, $\Pi x(Rx \to ((\neg\mathcal{A}_xB \wedge \mathcal{W}_x(A \to B)) \to \neg\mathcal{W}_xA))$, $\Pi x(Rx \to (\mathcal{W}_x(A \to B) \to (\mathcal{W}_x\neg B \to \neg\mathcal{W}_xA)))$, $\Pi x(Rx \to ((\mathcal{W}_x\neg B \wedge \mathcal{W}_x(A \to B)) \to \neg\mathcal{W}_xA))$, $\Pi x(Rx \to (\mathcal{W}_xB \to \neg\mathcal{R}_xB))$, $\Pi x(Rx \to (\mathcal{R}_xB \to \neg\mathcal{W}_xB))$, $\Pi x(Rx \to \neg(\mathcal{R}_xB \wedge \mathcal{R}_x\neg B))$, $\Pi x(Rx \to \neg(\mathcal{W}_xB \wedge \mathcal{R}_xB))$, $\Pi x(Rx \to \neg(\mathcal{W}_x(A \vee B) \wedge (\mathcal{R}_xA \wedge \mathcal{R}_xB)))$, $\Pi x(Rx \to (\mathcal{W}_x(A \to B) \to (\mathcal{W}_xA \to \neg\mathcal{R}_xB)))$, $\Pi x(Rx \to ((\mathcal{W}_xA \wedge \mathcal{W}_x(A \to B)) \to \neg\mathcal{R}_xB))$, $\Pi x(Rx \to (\mathcal{W}_x(A \to B) \to (\mathcal{R}_xB \to \neg\mathcal{W}_xA)))$, $\Pi x(Rx \to ((\mathcal{R}_xB \wedge \mathcal{W}_x(A \to B)) \to \neg\mathcal{W}_xA))$.

Some 'alethic-boulesic' sentences that are theorems in every $\Box\mathcal{W}$-system. All the following sentences (schemas) are theorems in every $\Box\mathcal{W}$-system in this paper: $\Pi x(Rx \to (\Box(A \to B) \to (\mathcal{W}_xA \to \mathcal{W}_xB)))$, $\Pi x(Rx \to ((\mathcal{W}_xA \wedge \Box(A \to B)) \to \mathcal{W}_xB))$, $\Pi x(Rx \to (\Box(A \to B) \to (\neg\mathcal{W}_xB \to \neg\mathcal{W}_xA)))$, $\Pi x(Rx \to ((\neg\mathcal{W}_xB \wedge \Box(A \to B)) \to \neg\mathcal{W}_xA))$, $\Pi x(Rx \to (\Box(A \to B) \to (\mathcal{W}_x\neg B \to \mathcal{W}_x\neg A)))$, $\Pi x(Rx \to ((\mathcal{W}_x\neg B \wedge \Box(A \to B)) \to \mathcal{W}_x\neg A))$, $\Pi x(Rx \to (\Box(A \to B) \to (\mathcal{A}_xA \to \mathcal{A}_xB)))$, $\Pi x(Rx \to$

$((\mathcal{A}_x A \wedge \Box(A \to B)) \to \mathcal{A}_x B))$, $\Pi x(Rx \to (\Box(A \to B) \to (\neg\mathcal{A}_x B \to \neg\mathcal{A}_x A)))$, $\Pi x(Rx \to ((\neg\mathcal{A}_x B \wedge \Box(A \to B)) \to \neg\mathcal{A}_x A))$, $\Pi x(Rx \to (\Box(A \to B) \to (\mathcal{A}_x \neg B \to \mathcal{A}_x \neg A)))$, $\Pi x(Rx \to ((\mathcal{A}_x \neg B \wedge \Box(A \to B)) \to \mathcal{A}_x \neg A))$, $\Pi x(Rx \to (\Box(A \leftrightarrow B) \to (\mathcal{W}_x A \leftrightarrow \mathcal{W}_x B)))$, $\Pi x(Rx \to (\Box(A \leftrightarrow B) \to (\neg\mathcal{W}_x A \leftrightarrow \neg\mathcal{W}_x B)))$, $\Pi x(Rx \to (\Box(A \leftrightarrow B) \to (\mathcal{W}_x \neg A \leftrightarrow \mathcal{W}_x \neg B)))$, $\Pi x(Rx \to (\Box(A \leftrightarrow B) \to (\mathcal{A}_x A \leftrightarrow \mathcal{A}_x B)))$, $\Pi x(Rx \to (\Box(A \leftrightarrow B) \to (\neg\mathcal{A}_x A \leftrightarrow \neg\mathcal{A}_x B)))$, $\Pi x(Rx \to (\Box(A \leftrightarrow B) \to (\mathcal{A}_x \neg A \leftrightarrow \mathcal{A}_x \neg B)))$, $\Pi x(Rx \to (\Box(A \to B) \to (\mathcal{R}_x B \to \mathcal{R}_x A)))$, $\Pi x(Rx \to ((\mathcal{R}_x B \wedge \Box(A \to B)) \to \mathcal{R}_x A))$, $\Pi x(Rx \to (\Box(A \to B) \to (\neg\mathcal{R}_x A \to \neg\mathcal{R}_x B)))$, $\Pi x(Rx \to ((\neg\mathcal{R}_x A \wedge \Box(A \to B)) \to \neg\mathcal{R}_x B))$, $\Pi x(Rx \to (\Box(A \to B) \to (\mathcal{R}_x \neg A \to \mathcal{R}_x \neg B)))$, $\Pi x(Rx \to ((\mathcal{R}_x \neg A \wedge \Box(A \to B)) \to \mathcal{R}_x \neg B))$, $\Pi x(Rx \to (\Box(A \leftrightarrow B) \to (\mathcal{R}_x A \leftrightarrow \mathcal{R}_x B)))$, $\Pi x(Rx \to (\Box(A \leftrightarrow B) \to (\neg\mathcal{R}_x A \leftrightarrow \neg\mathcal{R}_x B)))$, $\Pi x(Rx \to (\Box(A \leftrightarrow B) \to (\mathcal{R}_x \neg A \leftrightarrow \mathcal{R}_x \neg B)))$, $\Pi x(Rx \to (\Box(A \leftrightarrow B) \to (\mathcal{I}_x A \leftrightarrow \mathcal{I}_x B)))$, $\Pi x(Rx \to (\Box(A \leftrightarrow B) \to (\neg\mathcal{I}_x A \leftrightarrow \neg\mathcal{I}_x B)))$, $\Pi x(Rx \to (\Box(A \leftrightarrow B) \to (\mathcal{I}_x \neg A \leftrightarrow \mathcal{I}_x \neg B)))$, $\Pi x(Rx \to (\Box(A \leftrightarrow B) \to (\mathcal{N}_x A \leftrightarrow \mathcal{N}_x B)))$, $\Pi x(Rx \to (\Box(A \leftrightarrow B) \to (\neg\mathcal{N}_x A \leftrightarrow \neg\mathcal{N}_x B)))$, $\Pi x(Rx \to (\Box(A \leftrightarrow B) \to (\mathcal{N}_x \neg A \leftrightarrow \mathcal{N}_x \neg B)))$.

Some 'alethic-boulesic' sentences that are theorems in every system that includes $\Box\mathcal{W}$ and bD (or $\mathcal{W}\Diamond$). $\Pi x(Rx \to (\Box(A \to B) \to (\mathcal{W}_x A \to \mathcal{A}_x B)))$, $\Pi x(Rx \to ((\mathcal{W}_x A \wedge \Box(A \to B)) \to \mathcal{A}_x B))$, $\Pi x(Rx \to (\Box(A \to B) \to (\neg\mathcal{A}_x B \to \neg\mathcal{W}_x A)))$, $\Pi x(Rx \to ((\neg\mathcal{A}_x B \wedge \Box(A \to B)) \to \neg\mathcal{W}_x A))$, $\Pi x(Rx \to (\Box(A \to B) \to (\mathcal{W}_x \neg B \to \neg\mathcal{W}_x A)))$, $\Pi x(Rx \to ((\mathcal{W}_x \neg B \wedge \Box(A \to B)) \to \neg\mathcal{W}_x A))$, $\Pi x(Rx \to \neg(\Box(A \vee B) \wedge (\mathcal{W}_x \neg A \wedge \mathcal{W}_x \neg B)))$, $\Pi x(Rx \to \neg(\Box(A \vee B) \wedge (\mathcal{R}_x A \wedge \mathcal{R}_x B)))$, $\Pi x(Rx \to (\Box(A \to B) \to (\mathcal{W}_x A \to \neg\mathcal{R}_x B)))$, $\Pi x(Rx \to ((\mathcal{W}_x A \wedge \Box(A \to B)) \to \neg\mathcal{R}_x B))$, $\Pi x(Rx \to (\Box(A \to B) \to (\mathcal{R}_x B \to \neg\mathcal{W}_x A)))$, $\Pi x(Rx \to ((\mathcal{R}_x B \wedge \Box(A \to B)) \to \neg\mathcal{W}_x A))$.

Some 'deontic boulesic' sentences that are theorems in every system that includes $T - \mathbf{O}\mathcal{W}$ (and the definitions in Definition 1). $\Pi x(Rx \to (\mathbf{O}A \to \mathcal{W}_x A))$, $\Pi x(Rx \to (\mathbf{F}A \to \mathcal{R}_x A))$, $\Pi x(Rx \to (\mathcal{A}_x B \to \mathbf{P}B))$, $\mathbf{O}A \to \Pi x(Rx \to \mathcal{W}_x A)$, $\mathbf{F}A \to \Pi x(Rx \to \mathcal{R}_x A)$, $\Sigma x Rx \to (\Pi x(Rx \to \mathcal{A}_x B) \to \mathbf{P}B)$, $\Sigma x(Rx \wedge \mathcal{A}_x B) \to \mathbf{P}B$, $\Sigma x Rx \to (\mathbf{O}A \to \Sigma x(Rx \wedge \mathcal{W}_x A))$, $\Sigma x Rx \to (\mathbf{F}A \to \Sigma x(Rx \wedge \mathcal{R}_x A))$. The sentence $\mathbb{U}\Pi x(Rx \to (\mathbf{O}A \to \mathcal{W}_x A))$, which is a theorem in every $T - \mathbf{O}\mathcal{W}$-system, says that it is absolutely necessary that for every individual x, if x is perfectly rational, then if it it ought to be the case that A, then x wants it to be the case that A. This is a version of a philosophically very interesting principle often called 'internalism' (see Introduction). More precisely, it is a kind of conditional existence internalism. The following sentence is an instance of this schema: 'it is absolutely necessary that if the individual c is perfectly rational, then if c ought to do the action H, then c wants to do H'. Nevertheless, if c is not perfectly rational, it is not necessary that she wants to do H. Accordingly, this kind of internalism is compatible with the existence of amoralists and with the phenomenon of weakness of will. $\mathbf{O}A \to \Pi x(Rx \to \mathcal{W}_x A)$ is similar. It says that if it ought to be the case that A, then everyone who is perfectly rational wants it to be the case that A. The other theorems mentioned in this paragraph are also closely connected to the theory of internalism.[9]

Some 'boulesic deontic' sentences that are theorems in every system that includes $T - \mathcal{W}\mathbf{O}$ (and the definitions in Definition 1). $\Pi x(Rx \to (\mathbf{P}B \to \mathcal{A}_x B))$,

[9]For more on internalism and various interpretations of this thesis and for an introduction to some arguments for and against it, see, for example, [23], [24] and [151].

$\Pi x(Rx \to (\mathcal{W}_x A \to \mathbf{O}A))$, $\Pi x(Rx \to (\mathcal{R}_x A \to \mathbf{F}A))$, $\mathbf{P}B \to \Pi x(Rx \to \mathcal{A}_x B)$, $\Sigma x Rx \to (\Pi x(Rx \to \mathcal{W}_x A) \to \mathbf{O}A)$, $\Sigma x Rx \to (\Pi x(Rx \to \mathcal{R}_x A) \to \mathbf{F}A)$, $\Sigma x(Rx \wedge \mathcal{W}_x A) \to \mathbf{O}A$, $\Sigma x(Rx \wedge \mathcal{R}_x A) \to \mathbf{F}A$, $\Sigma x Rx \to (\mathbf{P}B \to \Sigma x(Rx \wedge \mathcal{A}_x B))$. The converse of $\mathbf{O}A \to \Pi x(Rx \to \mathcal{W}_x A)$ is $\Pi x(Rx \to \mathcal{W}_x A) \to \mathbf{O}A$. We cannot prove the latter formula. However, we can establish something similar in every $T-\mathcal{W}\mathbf{O}$-system, namely the following principle: $\Sigma x Rx \to (\Pi x(Rx \to \mathcal{W}_x A) \to \mathbf{O}A)$. This theorem says that if there is someone who is perfectly rational, then if everyone who is perfectly rational wants it to be the case that A then it ought to be the case that A. $\Pi x(Rx \to (\mathcal{W}_x A \to \mathbf{O}A))$ says that if x is perfectly rational, then if x wants it to be the case that A then it ought to be the case that A (to prove this theorem we do not have to assume that there is someone who is perfectly rational). If no one is perfectly rational, then $\Sigma x Rx \to (\Pi x(Rx \to \mathcal{W}_x A) \to \mathbf{O}A)$ is vacuously true. $\mathbf{P}B \to \Pi x(Rx \to \mathcal{A}_x B)$ says that it is permitted that B only if everyone who is perfectly rational accepts B (if it is permitted that B then everyone who is perfectly rational accepts B).

Some 'deontic boulesic' sentences that are theorems in every system that includes $T - \mathbf{O}\mathcal{A}$. $\Pi x(Rx \to (\mathbf{O}B \to \mathcal{A}_x B))$, $\Pi x(Rx \to (\mathcal{W}_x B \to \mathbf{P}B))$, $\mathbf{O}B \to \Pi x(Rx \to \mathcal{A}_x B)$, $\Sigma x Rx \to (\Pi x(Rx \to \mathcal{W}_x A) \to \mathbf{P}A)$. $\mathbf{O}B \to \Pi x(Rx \to \mathcal{A}_x B)$ says that it ought to be the case that B only if everyone who is perfectly rational accepts B (if it ought to be the case that B then everyone who is perfectly rational accepts that it is the case that B). $\Pi x(Rx \to (\mathbf{O}B \to \mathcal{A}_x B))$ says that if x is perfectly rational, then if it ought to be the case that B then x accepts that it is the case that B. $\Pi x(Rx \to (\mathcal{W}_x B \to \mathbf{P}B))$ says that if x is perfectly rational then x wants B only if it is permitted that B.

Some 'boulesic deontic' sentences that are theorems in every system that includes $T - \mathbf{O}\mathcal{W}$ and $T - \mathcal{W}\mathbf{O}$ (and the definitions in Definition 1). $\Pi x(Rx \to (\mathbf{O}A \leftrightarrow \mathcal{W}_x A))$, $\Pi x(Rx \to (\mathbf{F}A \leftrightarrow \mathcal{R}_x A))$, $\Pi x(Rx \to (\mathbf{P}B \leftrightarrow \mathcal{A}_x B))$, $\Pi x(Rx \to (\mathcal{W}_x A \leftrightarrow \mathbf{O}A))$, $\Pi x(Rx \to (\mathcal{R}_x A \leftrightarrow \mathbf{F}A))$, $\Pi x(Rx \to (\mathcal{A}_x B \leftrightarrow \mathbf{P}B))$, $\Sigma x Rx \to (\mathbf{O}A \leftrightarrow \Pi x(Rx \to \mathcal{W}_x A))$, $\Sigma x Rx \to (\mathbf{F}A \leftrightarrow \Pi x(Rx \to \mathcal{R}_x A))$, $\Sigma x Rx \to (\mathbf{P}B \leftrightarrow \Pi x(Rx \to \mathcal{A}_x B))$, $\Sigma x Rx \to (\mathbf{O}A \leftrightarrow \Sigma x(Rx \wedge \mathcal{W}_x A))$, $\Sigma x Rx \to (\mathbf{F}A \leftrightarrow \Sigma x(Rx \wedge \mathcal{R}_x A))$, $\Sigma x Rx \to (\mathbf{P}B \leftrightarrow \Sigma x(Rx \wedge \mathcal{A}_x B))$. $\Pi x(Rx \to (\mathbf{O}A \leftrightarrow \mathcal{W}_x A))$ says that if x is perfectly rational then it ought to be the case that A iff x wants it to be the case that A. This is a principle that a Kantian might want to include in his or her system (see Introduction). $\Pi x(Rx \to (\mathcal{A}_x B \leftrightarrow \mathbf{P}B))$ says that if x is perfectly rational then x accepts that it is the case that B iff it is permitted that B, etc. So, suppose that x is perfectly rational. Then, it ought to be the case that A iff x wants it to be the case that A (and x wants it to be the case that A iff it ought to be the case that A), it is forbidden (wrong) that A iff x rejects A (and x rejects A iff it is forbidden (wrong) that A), it is permitted (right) that A iff x accepts that it is the case that A (and x accepts A iff it is permitted (right) that A), etc. Hence, in every system that includes $T - \mathbf{O}\mathcal{W}$ and $T - \mathcal{W}\mathbf{O}$ (and the definitions in Definition 1), we can show that every perfectly rational individual has boulesic attitudes that are perfectly aligned with all the norms (all obligations, permissions and prohibitions). $\Sigma x Rx \to (\mathbf{O}A \leftrightarrow \Pi x(Rx \to \mathcal{W}_x A))$ says that if there is someone who is perfectly rational, then it ought to be the case that A iff everyone who is perfectly rational wants it to be the case that A; $\Sigma x Rx \to (\mathbf{P}B \leftrightarrow \Pi x(Rx \to \mathcal{A}_x B))$ says that if there is someone who is perfectly rational then it is permitted that B iff everyone who is perfectly rational accepts that it is the case that B, etc. Suppose that there is someone who is perfectly rational. Then, if our system includes $T - \mathbf{O}\mathcal{W}$ and $T - \mathcal{W}\mathbf{O}$, we can prove the following equivalences: it ought to be the case that A iff everyone who is perfectly rational wants it to be the case that A; it is permitted that A iff everyone who is perfectly rational accepts that it is the case that A; and it is forbidden that A iff everyone who is perfectly rational rejects A. Accordingly (if we assume that there is someone who is perfectly rational), the theorems in this paragraph can be interpreted as a kind of ideal observer theory for normative propositions.[10] If a system includes $T-\mathbf{O}\mathcal{W}$ and $T-\mathcal{W}\mathbf{O}$ we can also prove the following formulas: $\Pi x\Pi y((Rx \wedge Ry) \to (\mathcal{W}_x B \to \mathcal{W}_y B))$, $\Pi x(Rx \to (\mathcal{W}_x B \to \Pi y(Ry \to \mathcal{W}_y B))$, $\Pi x\Pi y((Rx \wedge Ry) \to (\mathcal{A}_x B \to \mathcal{A}_y B))$, $\Pi x(Rx \to (\mathcal{A}_x B \to \Pi y(Ry \to \mathcal{A}_y B)))$, $\Pi x\Pi y((Rx \wedge Ry) \to (\mathcal{R}_x B \to \mathcal{R}_y B))$, $\Pi x(Rx \to (\mathcal{R}_x B \to \Pi y(Ry \to \mathcal{R}_y B))$, $\Sigma x(Rx \wedge \mathcal{W}_x B) \to \Pi x(Rx \to \mathcal{W}_x B)$, $\Sigma x(Rx \wedge \mathcal{A}_x B) \to \Pi x(Rx \to \mathcal{A}_x B)$, $\Sigma x(Rx \wedge \mathcal{R}_x B) \to \Pi x(Rx \to \mathcal{R}_x B)$. $\Pi x\Pi y((Rx \wedge Ry) \to (\mathcal{W}_x B \to \mathcal{W}_y B))$ says that if x is perfectly rational and y is perfectly rational, then if x wants it to be the case that B then y wants it to be the case that B; $\Pi x\Pi y((Rx \wedge Ry) \to (\mathcal{A}_x B \to \mathcal{A}_y B))$ says that if x is perfectly rational and y is perfectly rational, then if x accepts that it is the case that B then y accepts that it is the case that B, etc. So, if a system includes $T-\mathbf{O}\mathcal{W}$ and $T-\mathcal{W}\mathbf{O}$ we can prove that all perfectly rational individuals want, accept and reject the same things.[11]

Some 'boulesic' and 'alethic boulesic' sentences that are theorems in various systems. In every system that includes $T-b4$, $\Pi x((Rx \wedge \mathcal{W}_x Rx) \to (\mathcal{W}_x B \to \mathcal{W}_x \mathcal{W}_x B))$ is a theorem. In every system that includes $T-b5$, $\Pi x((Rx \wedge \mathcal{W}_x Rx) \to (\mathcal{A}_x B \to \mathcal{W}_x \mathcal{A}_x B))$ is a theorem. In every system that includes $T-\mathcal{W}bT$, $\Pi x((Rx \wedge \mathcal{W}_x Rx) \to \mathcal{W}_x(\mathcal{W}_x B \to B))$ is a theorem. In every system that includes $T-\mathcal{W}bB$ and $b4$, $\Pi x((Rx \wedge \mathcal{W}_x Rx) \to \mathcal{W}_x(\mathcal{A}_x \mathcal{W}_x A \to A))$ is a theorem. $\Pi x((Rx \wedge \mathcal{W}_x Rx) \to \mathcal{W}_x(\Box A \to \mathcal{W}_x A))$ is a theorem in every system that includes $T-\mathcal{W}\Box\mathcal{W}$ and $\Pi x((Rx \wedge \mathcal{W}_x Rx) \to \mathcal{W}_x(\mathcal{W}_x A \to \Diamond A))$ is a theorem in every system that includes $T-\mathcal{W}\mathcal{W}\Diamond$.

Barcan-like formulas. The following Barcan-like formulas can be proved in every system in this paper: $\Pi x(Rx \to (\Pi y \mathcal{W}_x B \leftrightarrow \mathcal{W}_x \Pi y B))$, $\Pi x(Rx \to (\Sigma y \mathcal{A}_x B \leftrightarrow \mathcal{A}_x \Sigma y B))$, $\Pi x(Rx \to (\mathcal{A}_x \Pi y B \to \Pi y \mathcal{A}_x B))$, and $\Pi x(Rx \to (\Sigma y \mathcal{W}_x B \to \mathcal{W}_x \Sigma y B))$. Let $\blacksquare$ be $\mathbb{U}$, $\Box$, $\mathbb{A}$, $\mathbb{G}$, $\mathbb{H}$, $\mathbf{O}$, $\underline{\mathbb{G}}$ or $\underline{\mathbb{H}}$, and let $\blacklozenge$ be the dual of $\blacksquare$. Then, all the following sentences (schemas) are theorems in every system in this paper: $\Pi x \blacksquare B \leftrightarrow \blacksquare \Pi x B$, $\Sigma x \blacklozenge B \leftrightarrow \blacklozenge \Sigma x B$, $\blacklozenge \Pi x B \to \Pi x \blacklozenge B$, and $\Sigma x \blacksquare B \to \blacksquare \Sigma x B$.

Some theorems that can be proved in systems that include the transfer rules. In every system that includes $T-UR$ or $T-FTR$ and $T-\Box\mathcal{W}$, we can prove that the following sentence is a theorem: $\Pi x(Rx \to \mathcal{W}_x Rx)$, which says that everyone who is perfectly rational wants to be perfectly rational. In every system that includes $T-UR$ or $T-FTR$ and $T-\Box\mathcal{W}$, and $T-bD$, we can prove that the following sentence is a theorem: $\Pi x(Rx \to \mathcal{A}_x Rx)$, which says that everyone who is perfectly rational accepts that she is perfectly rational. In every system that includes $T-UR$, we can

[10] For more on ideal observer theories, see, for example, [56] and [90]. I will not try to decide whether or not it is reasonable to assume that there is someone who is perfectly rational in this paper. However, note that our systems do not exclude that there are things that do not exist and that non-existing things have properties. So, the statement that there is someone who is perfectly rational does not necessarily entail that this individual exists. If being perfectly rational is not an existence-entailing property, our systems are compatible with the proposition that there are non-existing perfectly rational individuals.

[11] This does not entail that every perfectly rational individual wants every individual to do the same things and have the same properties. For example, the view is compatible with the proposition that both c and d want e to be P and that both c and d want f to be not-P.

prove the following sentence: $\Pi x(Rx \to \mathbb{U}Rx)$, which says that every perfectly rational individual is necessarily perfectly rational.

Some theorems that include the identity sign. Let $\mathcal{O}$ be a boulesic operator ($\mathcal{W}$, $\mathcal{A}$, $\mathcal{R}$, $\mathcal{I}$ or $\mathcal{N}$). Then, if a system includes ($T - S$ =) and ($T - A$ =), we can prove the following theorems in this system: $(\mathcal{O}_c B \land c = d) \to \mathcal{O}_d B$ and $\Pi x \Pi y((\mathcal{O}_x B \land x = y) \to \mathcal{O}_y B)$, which are intuitively plausible. By using ($T - N$ =), we can show that all identities and non-identities are (absolutely and historically) necessary and eternal, that is, we can prove all the following theorems: $\Pi x \Pi y(x = y \to \mathbb{U}x = y)$, $\Pi x \Pi y(x = y \to \Box x = y)$, $\Pi x \Pi y(\neg x = y \to \mathbb{U}\neg x = y)$, $\Pi x \Pi y(\neg x = y \to \Box\neg x = y)$, $\Pi x \Pi y(x = y \to \mathbb{A}x = y)$ and $\Pi x \Pi y(\neg x = y \to \mathbb{A}\neg x = y)$. Since every constant is treated as a rigid designator in this paper, this is plausible.

Some theorems that include temporal and boulesic operators. In every system that includes $T - \mathcal{W}\mathbb{G}bT$, $\Pi x((Rx \land \mathcal{W}_x \mathbb{G}Rx) \to \mathcal{W}_x \mathbb{G}(\mathcal{W}_x A \to A))$ is a theorem and $\Pi x(Rx \to \mathcal{W}_x \mathbb{G}(\mathcal{W}_x A \to A))$ can be proved in every system that includes $T - \mathcal{W}\mathbb{G}bT$ and $T - UR$. $\Pi x((Rx \land \mathcal{W}_x \mathbb{G}Rx \land \mathcal{W}_x \mathbb{G}\mathcal{W}_x Rx) \to \mathcal{W}_x \mathbb{G}(B \to \mathcal{W}_x \mathcal{A}_x B))$ is a theorem in every system that contains $T - \mathcal{W}\mathbb{G}bB$ and $\Pi x(Rx \to \mathcal{W}_x \mathbb{G}(B \to \mathcal{W}_x \mathcal{A}_x B))$ is a theorem in every system that contains $T - \mathcal{W}\mathbb{G}bB$ and $T - UR$. $\Pi x((Rx \land \mathcal{W}_x \mathbb{G}Rx) \to (\mathcal{W}_x \mathbb{G}A \to \mathcal{W}_x \mathbb{G}\mathcal{W}_x A))$ can be proved in every system that includes $T - BR$.

Some theorems in various systems. Let A be a formula in Section 3.3. Then if A is valid in every model that satisfies the semantic conditions $C - X_1, \ldots, C - X_n$, then A is a theorem in every quantified temporal alethic boulesic deontic tableau system that includes the tableau rules $T - X_1, \ldots, T - X_n$. We observed that if a model satisfies $C - bD$, then $\Pi x(Rx \to \neg(\mathcal{W}_x B \land \mathcal{W}_x \neg B))$ is valid in this model. Hence, $\Pi x(Rx \to \neg(\mathcal{W}_x B \land \mathcal{W}_x \neg B))$ is a theorem in every quantified temporal alethic boulesic deontic tableau system that includes $T - bD$. We observed that if a model satisfies $C - \mathbf{O}\mathbb{G}\Box\mathcal{W}$ and $C - UR$, $\Pi x(Rx \to \mathbf{O}\mathbb{G}(\Box A \to \mathcal{W}_x A))$ is valid in this model. Hence, $\Pi x(Rx \to \mathbf{O}\mathbb{G}(\Box A \to \mathcal{W}_x A))$ is a theorem in every quantified temporal alethic boulesic deontic tableau system that includes $T - \mathbf{O}\mathbb{G}\Box\mathcal{W}$ and $T - UR$, etc.

6 Soundness and completeness theorems

This section establishes the soundness and completeness of every system in this paper. Let us begin by defining these concepts.

Definition 8 ***(Soundness and completeness)*** *Let $S = T - A_1, ..., T - A_n$ be a quantified temporal alethic boulesic deontic tableau system as defined in Section 4 (Definition 5) above (where $T - A_1, ..., T - A_n$ are the non-basic tableau rules in S). Then we shall say that the class of models, $\boldsymbol{M}$, corresponds to S iff $\boldsymbol{M} = \boldsymbol{M}(C - A_1, ..., C - A_n)$.*

S is sound with respect to $\boldsymbol{M}$ iff $\Gamma \vdash_S A$ entails $\boldsymbol{M}, \Gamma \Vdash A$, and S is complete with respect to $\boldsymbol{M}$ just in case $\boldsymbol{M}, \Gamma \Vdash A$ entails $\Gamma \vdash_S A$ (where $\boldsymbol{M}$ corresponds to S).

Lemma 9 ***(Locality)***
Let $\mathcal{M}_1 = \langle D, W, T, <, \mathfrak{R}, \mathfrak{A}, \mathfrak{S}, v_1 \rangle$ and $\mathcal{M}_2 = \langle D, W, T, <, \mathfrak{R}, \mathfrak{A}, \mathfrak{S}, v_2 \rangle$ be two models. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same domain, the language of $\mathcal{M}_1$ is the same as the language of $\mathcal{M}_2$. Let us call this language $\mathcal{L}$. Moreover, let A be any closed formula

of $\mathcal{L}$ such that v_1 and v_2 agree on the denotations of all the predicates, constants and matrices in it. Then for all $\omega \in W$ and $\tau \in T$: $\mathcal{M}_1, \omega, \tau \Vdash A$ iff $\mathcal{M}_2, \omega, \tau \Vdash A$.

Proof. The result is established by recursion on formulas; 'the IH' is an abbreviation of 'the induction hypothesis'.

Atomic formulas. $\mathcal{M}_1, \omega, \tau \Vdash Pa_1 \ldots a_n$ iff $\langle v_1(a_1), \ldots, v_1(a_n)\rangle \in v_{1\omega\tau}(P)$ iff $\langle v_2(a_1), \ldots, v_2(a_n)\rangle \in v_{2\omega\tau}(P)$ iff $\mathcal{M}_2, \omega, \tau \Vdash Pa_1 \ldots a_n$.

Suppose that $\mathcal{M}_1, \omega, \tau \nVdash Ra_m$, that M is a matrix where x_m is the first free variable in M and that a_m is the constant in $M[a_1, \ldots, a_n/\vec{x}]$ that replaces x_m. Then: $\mathcal{M}_2, \omega, \tau \nVdash Ra_m$ and $\mathcal{M}_1, \omega, \tau \Vdash M[a_1, \ldots, a_n/\vec{x}]$ iff $\langle v_1(a_1), \ldots, v_1(a_n)\rangle \in v_{1\omega\tau}(M)$ iff $\langle v_2(a_1), \ldots, v_2(a_n)\rangle \in v_{2\omega\tau}(M)$ iff $\mathcal{M}_2, \omega, \tau \Vdash M[a_1, \ldots, a_n/\vec{x}]$.

Truth-functional connectives. Straightforward.

($\Box$). $\mathcal{M}_1, \omega, \tau \Vdash \Box B$ iff for all ω' such that $\mathfrak{R}\omega\omega'\tau$, $\mathcal{M}_1, \omega', \tau \Vdash B$ iff for all ω' such that $\mathfrak{R}\omega\omega'\tau$, $\mathcal{M}_2, \omega', \tau \Vdash B$ [the IH] iff $\mathcal{M}_2, \omega, \tau \Vdash \Box B$.

Other alethic, temporal and deontic operators. Similar.

($\mathcal{W}_c D$). A is of the form $\mathcal{W}_c D$. Assume that $\mathcal{M}_1, \omega, \tau \Vdash \mathcal{W}_c D$. We have two cases: $\mathcal{M}_1, \omega, \tau \nVdash Rc$ or $\mathcal{M}_1, \omega, \tau \Vdash Rc$. Suppose $\mathcal{M}_1, \omega, \tau \nVdash Rc$. Then $\mathcal{M}_2, \omega, \tau \nVdash Rc$. Hence, $\mathcal{M}_2, \omega, \tau \Vdash \mathcal{W}_c D$. And vice versa. Suppose $\mathcal{M}_1, \omega, \tau \Vdash Rc$. Then for all ω' such that $\mathfrak{A}v_1(c)\omega\omega'\tau$: $\mathcal{M}_1, \omega', \tau \Vdash D$. Accordingly, for all ω' such that $\mathfrak{A}v_2(c)\omega\omega'\tau$: $\mathcal{M}_2, \omega', \tau \Vdash D$ [by assumption and the IH]. Moreover, $\mathcal{M}_2, \omega, \tau \Vdash Rc$. Hence, $\mathcal{M}_2, \omega, \tau \Vdash \mathcal{W}_c D$. And vice versa. It follows that $\mathcal{M}_1, \omega, \tau \Vdash \mathcal{W}_c D$ iff $\mathcal{M}_2, \omega, \tau \Vdash \mathcal{W}_c D$.

Other boulesic operators. Similar.

(Π). $\mathcal{M}_1, \omega, \tau \Vdash \Pi x B$ iff for all $k_d \in \mathcal{L}$, $\mathcal{M}_1, \omega, \tau \Vdash B[k_d/x]$ iff for all $k_d \in \mathcal{L}$, $\mathcal{M}_2, \omega, \tau \Vdash B[k_d/x]$ [by the IH, and the fact that $v_{1\omega\tau}(k_d) = v_{2\omega\tau}(k_d) = d$] iff $\mathcal{M}_2, \omega, \tau \Vdash \Pi x B$.

The particular quantifier. Similar. ∎

Lemma 10 ***(Denotation)*** *Let $\mathcal{M} = \langle D, W, T, <, \mathfrak{R}, \mathfrak{A}, \mathfrak{S}, v\rangle$ be any model. Let A be any formula of the language of $\mathcal{M}$ ($\mathcal{L}(\mathcal{M})$) with at most one free variable, x, and a and b be any two constants such that $v(a) = v(b)$. Then for any $\omega \in W$ and $\tau \in T$: $\mathcal{M}, \omega, \tau \Vdash A[a/x]$ iff $\mathcal{M}, \omega, \tau \Vdash A[b/x]$.*

Proof. The proof is by recursion on sentences.

Atomic formulas. (To illustrate, we assume that the formula has one occurrence of 'a' distinct from each a_i.) $\mathcal{M}, \omega, \tau \Vdash Pa_1 \ldots a \ldots a_n$ iff $\langle v(a_1), \ldots, v(a), \ldots, v(a_n)\rangle \in v_{\omega\tau}(P)$ iff $\langle v(a_1), \ldots, v(b), \ldots, v(a_n)\rangle \in v_{\omega\tau}(P)$ iff $\mathcal{M}, \omega, \tau \Vdash Pa_1 \ldots b \ldots a_n$.

Suppose $\mathcal{M}, \omega, \tau \nVdash Ra_m$, that M is a matrix where x_m is the first free variable in M and that a_m is the constant in $M[a_1, \ldots, a, \ldots, a_n/\vec{x}]$ ($M[a_1, \ldots, b, \ldots, a_n/\vec{x}]$) that replaces x_m. (To illustrate, we assume that the formula has one occurrence of 'a' distinct from each a_i and that a_m is not a (b).) Then: $\mathcal{M}, \omega, \tau \Vdash M[a_1, \ldots, a, \ldots, a_n/\vec{x}]$ iff $\langle v(a_1), \ldots, v(a), \ldots, v(a_n)\rangle \in v_{\omega\tau}(M)$ iff $\langle v(a_1), \ldots, v(b), \ldots, v(a_n)\rangle \in v_{\omega\tau}(M)$ iff $\mathcal{M}, \omega, \tau \Vdash M[a_1, \ldots, b, \ldots, a_n/\vec{x}]$.

Truth-functional connectives. Straightforward.

($\Box$). $\mathcal{M}, \omega, \tau \Vdash \Box B[a/x]$ iff for all ω' such that $\mathfrak{R}\omega\omega'\tau$, $\mathcal{M}, \omega', \tau \Vdash B[a/x]$ iff for all ω' such that $\mathfrak{R}\omega\omega'\tau$, $\mathcal{M}, \omega', \tau \Vdash B[b/x]$ [the IH] iff $\mathcal{M}, \omega, \tau \Vdash \Box B[b/x]$.

Other alethic, temporal and deontic operators. Similar.

($\mathcal{W}_t$). A is of the form $\mathcal{W}_t D$. Either $\mathcal{M}, \omega, \tau \Vdash Rt$ or $\mathcal{M}, \omega, \tau \nVdash Rt$. We have already established that the result holds if $\mathcal{M}, \omega, \tau \nVdash Rt$. So, suppose that $\mathcal{M}, \omega, \tau \Vdash Rt$. Since x is the only free variable, t cannot be a variable distinct from x. So, t is either x or a constant. Suppose t is x. Then $\mathcal{M}, \omega, \tau \Vdash \mathcal{W}_x D[a/x]$ iff $\mathcal{M}, \omega, \tau \Vdash \mathcal{W}_a D[a/x]$ iff for all ω' such that $\mathfrak{A}v(a)\omega\omega'\tau$, $\mathcal{M}, \omega', \tau \Vdash D[a/x]$ iff for all ω' such that $\mathfrak{A}v(b)\omega\omega'\tau$, $\mathcal{M}, \omega', \tau \Vdash D[b/x]$ [by the fact that $v(a) = v(b)$ and the IH] iff $\mathcal{M}, \omega, \tau \Vdash \mathcal{W}_b D[b/x]$ iff $\mathcal{M}, \omega, \tau \Vdash \mathcal{W}_x D[b/x]$. Suppose t is a constant, say c. Then $\mathcal{M}, \omega, \tau \Vdash \mathcal{W}_c D[a/x]$ iff for all ω' such that $\mathfrak{A}v(c)\omega\omega'\tau$, $\mathcal{M}, \omega', \tau \Vdash D[a/x]$ iff for all ω' such that $\mathfrak{A}v(c)\omega\omega'\tau$, $\mathcal{M}, \omega', \tau \Vdash D[b/x]$ [by the IH] iff $\mathcal{M}, \omega, \tau \Vdash \mathcal{W}_c D[b/x]$.

Other boulesic operators. Similar.

(Π). Let A be of the form $\Pi y B$. If $x = y$, then $A[a/x] = A[b/x] = A$, so the result is trivial. Hence, suppose that x and y are distinct. Then, $(\Pi y B)[b/x] = \Pi y(B[b/x])$ and $(B[b/x])[a/y] = (B[a/y])[b/x]$. $\mathcal{M}, \omega, \tau \Vdash (\Pi y B)[a/x]$ iff $\mathcal{M}, \omega, \tau \Vdash \Pi y(B[a/x])$ iff for all $k_d \in \mathcal{L}(\mathcal{M})$, $\mathcal{M}, \omega, \tau \Vdash (B[a/x])[k_d/y]$ iff for all $k_d \in \mathcal{L}(\mathcal{M})$, $\mathcal{M}, \omega, \tau \Vdash (B[k_d/y])[a/x]$ iff for all $k_d \in \mathcal{L}(\mathcal{M})$, $\mathcal{M}, \omega, \tau \Vdash (B[k_d/y])[b/x]$ [the IH] iff for all $k_d \in \mathcal{L}(\mathcal{M})$, $\mathcal{M}, \omega, \tau \Vdash (B[b/x])[k_d/y]$ iff $\mathcal{M}, \omega, \tau \Vdash \Pi y(B[b/x])$ iff $\mathcal{M}, \omega, \tau \Vdash (\Pi y B)[b/x]$.

The case for the particular quantifier (Σ) is similar. ■

6.1 Soundness theorem

Let $\mathcal{M} = \langle D, W, T, <, \mathfrak{R}, \mathfrak{A}, \mathfrak{S}, v\rangle$ be any model and $\mathcal{B}$ any branch of a tableau. Then $\mathcal{B}$ is satisfiable in $\mathcal{M}$ iff there is a function f from $w_0, w_1, w_2, \ldots$ to W, and a function g from $t_0, t_1, t_2, \ldots$ to T such that

(i) A is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$, for every node $A, w_i t_j$ on $\mathcal{B}$;

(ii) if $r w_i w_j t_k$ is on $\mathcal{B}$, then $\mathfrak{R} f(w_i) f(w_j) g(t_k)$ in $\mathcal{M}$;

(iii) if $A c w_i w_j t_k$ is on $\mathcal{B}$, then $\mathfrak{A}v(c) f(w_i) f(w_j) g(t_k)$ in $\mathcal{M}$;

(iv) if $s w_i w_j t_k$ is on $\mathcal{B}$, then $\mathfrak{S} f(w_i) f(w_j) g(t_k)$ in $\mathcal{M}$;

(v) if $t_i < t_j$ is on $\mathcal{B}$, then $g(t_i) < g(t_j)$ in $\mathcal{M}$;

(vi) if $t_i = t_j$ is on $\mathcal{B}$, then $g(t_i) = g(t_j)$ in $\mathcal{M}$.

If these conditions are fulfilled, we say that f and g show that $\mathcal{B}$ is satisfiable in $\mathcal{M}$.

Lemma 11 ***(Soundness Lemma)*** *Let $\mathcal{B}$ be any branch of a tableau and $\mathcal{M}$ be any model. If $\mathcal{B}$ is satisfiable in $\mathcal{M}$ and a tableau rule is applied to it, then there is a model $\mathcal{M}'$ and an extension of $\mathcal{B}$, $\mathcal{B}'$, such that $\mathcal{B}'$ is satisfiable in $\mathcal{M}'$.*

Proof. The proof is by induction on the height of the derivation. Let f and g be functions that show that the branch $\mathcal{B}$ is satisfiable in $\mathcal{M}$.

Connectives and modal, temporal and deontic operators. Straightforward.

($\mathcal{W}$). Suppose that $Rc, w_i t_k$, $\mathcal{W}_c D, w_i t_k$, and $A c w_i w_j t_k$ are on $\mathcal{B}$, and that we apply the $\mathcal{W}$-rule. Then we get an extension of $\mathcal{B}$ that includes $D, w_j t_k$. Since $\mathcal{B}$ is satisfiable in $\mathcal{M}$, $\mathcal{W}_c D$ and Rc are true in $f(w_i)$ at $g(t_k)$. Furthermore, for any w_i and w_j such that $A c w_i w_j t_k$ is on $\mathcal{B}$, $\mathfrak{A}v(c) f(w_i) f(w_j) g(t_k)$. Thus by the truth conditions for $\mathcal{W}_c D$, D is true in $f(w_j)$ at $g(t_k)$.

($\mathcal{A}$). Suppose that $Rc, w_i t_k$, $\mathcal{A}_c D, w_i t_k$ are on $\mathcal{B}$ and that we apply the $\mathcal{A}$-rule to get an extension of $\mathcal{B}$ that includes nodes of the form $Acw_i w_j t_k$ and $D, w_j t_k$. Since $\mathcal{B}$ is satisfiable in $\mathcal{M}$, $\mathcal{A}_c D$ and Rc are true in $f(w_i)$ at $g(t_k)$. Hence, for some ω in W, $\mathfrak{A}v(c) f(w_i)\omega g(t_k)$ and D is true in ω at $g(t_k)$ [by the truth conditions for $\mathcal{A}_c D$ and the fact that Rc is true in $f(w_i)$ at $g(t_k)$]. Let f' be the same as f except that $f'(w_j) = \omega$. Since f and f' differ only at w_j, f' and g show that $\mathcal{B}$ is satisfiable in $\mathcal{M}$. Furthermore, by definition $\mathfrak{A}v(c) f'(w_i) f'(w_j) g(t_k)$, and D is true in $f'(w_j)$ at $g(t_k)$.

Other boulesic operators. Similar.

(Π). Suppose that $\Pi x A, w_i t_j$ is on $\mathcal{B}$ and that we apply the Π-rule to get an extension of $\mathcal{B}$ that includes a node of the form $A[a/x], w_i t_j$. $\mathcal{M}$ makes $\Pi x A$ true in $f(w_i)$ at $g(t_j)$. For $\mathcal{B}$ is satisfiable in $\mathcal{M}$. Hence, $A[k_d/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$, for all $k_d \in \mathcal{L}(\mathcal{M})$. Let d be such that $v(a) = v(k_d)$. By the Denotation Lemma, $A[a/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Accordingly, we can take $\mathcal{M}'$ to be $\mathcal{M}$.

(Σ). Suppose that $\Sigma x A, w_i t_j$ is on $\mathcal{B}$ and that we apply the Σ-rule to get an extension of $\mathcal{B}$ that includes a node of the form $A[c/x], w_i t_j$ (where c is new). Since $\mathcal{B}$ is satisfiable in $\mathcal{M}$, $\Sigma x A$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}$. Consequently, there is some $k_d \in \mathcal{L}(\mathcal{M})$ such that $\mathcal{M}$ makes $A[k_d/x]$ true in $f(w_i)$ at $g(t_j)$. Let $\mathcal{M}' = \langle D, W, T, <, \mathfrak{R}, \mathfrak{A}, \mathfrak{S}, v' \rangle$ be the same as $\mathcal{M}$ except that $v'(c) = d$. Since c does not occur in $A[k_d/x]$, $A[k_d/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}'$, by the Locality Lemma. By the Denotation Lemma and the fact that $v'(c) = d = v'(k_d)$, $A[c/x]$ is true in $f(w_i)$ at $g(t_j)$ in $\mathcal{M}'$. Moreover, $\mathcal{M}'$ makes all other formulas on the branch true at their respective world-moment pairs as well, by the Locality Lemma. For c does not occur in any other formula on the branch.

($\neg\Pi$) and ($\neg\Sigma$). Straightforward.

Accessibility rules. I will consider three examples to illustrate the method.

($T - ab5$). Suppose that $rw_i w_j t_l$ and $Acw_i w_k t_l$ are on $\mathcal{B}$, and that we apply ($T - ab5$) to give an extended branch containing $Acw_j w_k t_l$. Since $\mathcal{B}$ is satisfiable in $\mathcal{M}$, $\mathfrak{R} f(w_i) f(w_j) g(t_l)$ and $\mathfrak{A}v(c) f(w_i) f(w_k) g(t_l)$. Hence, $\mathfrak{A}v(c) f(w_j) f(w_k) g(t_l)$ since $\mathcal{M}$ satisfies condition $C - ab5$. Consequently, the extension of $\mathcal{B}$ is satisfiable in $\mathcal{M}$.

($T - \mathcal{W}\mathbb{G}\mathbf{O}\mathcal{A}$). Suppose that $Acw_i w_j t_l$ and $t_l < t_m$ are on $\mathcal{B}$, and that we apply ($T - \mathcal{W}\mathbb{G}\mathbf{O}\mathcal{A}$) to give an extended branch containing $Acw_j w_k t_m$ and $sw_j w_k t_m$, where w_k is new. Since $\mathcal{B}$ is satisfiable in $\mathcal{M}$, $\mathfrak{A}v(c) f(w_i) f(w_j) g(t_l)$ and $g(t_l) < g(t_m)$. Hence, for some ω in W, $\mathfrak{A}v(c) f(w_j)\omega g(t_m)$ and $\mathfrak{S} f(w_j)\omega g(t_m)$, since $\mathcal{M}$ satisfies condition $C - \mathcal{W}\mathbb{G}\mathbf{O}\mathcal{A}$. Let f' be the same as f except that $f'(w_k) = \omega$. Since w_k does not occur on $\mathcal{B}$, f' and g show that $\mathcal{B}$ is satisfiable in $\mathcal{M}$. Moreover, $\mathfrak{A}v(c) f'(w_j) f'(w_k) g(t_m)$ and $\mathfrak{S} f'(w_j) f'(w_k) g(t_m)$ by construction. Hence, f' and g show that the extension of $\mathcal{B}$ is satisfiable in $\mathcal{M}$.

($T - \mathbf{O}\mathbb{G} \Box \mathbf{O}$). Suppose that $sw_i w_j t_l$, $t_l < t_m$ and $sw_j w_k t_m$ are on $\mathcal{B}$, and that we apply ($T - \mathbf{O}\mathbb{G} \Box \mathbf{O}$) to give an extended branch containing $rw_j w_k t_m$. Since $\mathcal{B}$ is satisfiable in $\mathcal{M}$, $\mathfrak{S} f(w_i) f(w_j) g(t_l)$, $g(t_l) < g(t_m)$ and $\mathfrak{S} f(w_j) f(w_k) g(t_m)$. Accordingly, $\mathfrak{R} f(w_j) f(w_k) g(t_m)$, for $\mathcal{M}$ satisfies condition $C - \mathbf{O}\mathbb{G}\Box\mathbf{O}$. In conclusion, the extension of $\mathcal{B}$ is satisfiable in $\mathcal{M}$. ■

Theorem 12 ***(Soundness Theorem)*** *Every system S in this paper is sound with respect to its semantics.*

Proof. Assume that B does not follow from Γ in $\boldsymbol{M}$, where $\boldsymbol{M}$ is the class of models that corresponds to S. Then every premise in Γ is true and the conclusion B false in some world ω at some time τ in some model $\mathcal{M}$ in $\boldsymbol{M}$. Consider an S-tableau whose first nodes consists of A, w_0t_0 for every $A \in \Gamma$ and $\neg B, w_0t_0$, where 'w_0' refers to ω and 't_0' refers to τ. The initial list in this tableau is satisfiable in $\mathcal{M}$. Every time we apply a rule to our tree it produces at least one extension that is satisfiable in a model $\mathcal{M}'$ in $\boldsymbol{M}$ (by the Soundness Lemma). Accordingly, we can find a whole branch such that every initial section of this branch is satisfiable in some model $\mathcal{M}''$ in $\boldsymbol{M}$. It is impossible that this branch is closed, for if it were closed, then some sentence would be both true and false in some possible world at some time in $\mathcal{M}''$. Therefore, the whole tableau is open. It follows that B is not derivable from Γ in S. Consequently, if B is derivable from Γ in S, then B follows from Γ in $\boldsymbol{M}$. ■

6.2 Completeness theorem

In this section, I will prove that every system in this paper is complete with respect to its semantics. First, I will define some important concepts.

We can think of a *complete* tableau as a tableau where every rule that can be applied has been applied. There can be several different (complete) tableaux for the same sentence or set of sentences, some more complex than others, for the tableau rules can be applied in different orders. To produce a complete tableau, we shall use the following method.[12] (1) For every open branch, one at a time, begin at its root and move towards its tip. Apply any rule that produces something *new* to the branch. If a rule has multiple applications (such as $\Box$ and Π), then make all possible applications at once. (2) Once we have done this for all open branches in the tableau, we repeat the procedure. Some rules, such as $T - aD$ and $T - \mathcal{W}\Diamond$ ($T - FD$), introduce new 'possible worlds' ('moments in time'). Every rule of this kind is applied once at the tip of every open branch at the end of every cycle (given that it produces something new). If a system includes more than one rule of this kind ($R1$, $R2$ ...), we alternate between them. The first time we use $R1$; the second time we use $R2$, etc. Before we conclude a cycle in this process we split the end of every open branch in the tree and add Rc, w_it_j to the left node and $\neg Rc, w_it_j$ to the right node, for every constant c (that occurs as an index to some boulesic operator on the tree), w_i and t_j on the branch. If there is still something to do according to this process, the tableau is incomplete; if not, it is complete.

Definition 13 ***(Induced Model)*** *Suppose that $\mathcal{B}$ is an open and complete branch of a tableau and that I is the set of numbers on $\mathcal{B}$ immediately preceded by a 't'. Let $i \rightleftharpoons j$ iff $i = j$, or '$t_i = t_j$' or '$t_j = t_i$' is on $\mathcal{B}$. $\rightleftharpoons$ is an equivalence relation and $[i]$ is the equivalence class of i. Furthermore, let C be the set of all constants on $\mathcal{B}$. We shall say that $a \sim b$ just in case $a = b, w_0t_0$ occurs on the branch. Obviously, $a \sim b$ is an equivalence relation. Let $[a]$ be the equivalence class of a under $\sim$. The model $\mathcal{M} = \langle D, W, T, <, \mathfrak{R}, \mathfrak{A}, \mathfrak{S}, v\rangle$ induced by $\mathcal{B}$ is defined as follows. $D = \{[a] : a \in C\}$ (or, if $C = \varnothing$, $D = \{o\}$ for an arbitrary o). (o is not in the extension of anything.) $W = \{\omega_i : w_i$ occurs on $\mathcal{B}\}$, $T = \{\tau_{[i]} : i \in I\}$, $\tau_{[i]} < \tau_{[j]}$ iff $t_i < t_j$ occurs on $\mathcal{B}$, $\mathfrak{R}\omega_i\omega_j\tau_{[k]}$ iff $rw_iw_jt_k$ occurs on $\mathcal{B}$, $\mathfrak{A}v(a)\omega_i\omega_j\tau_{[k]}$ iff $Aaw_iw_jt_k$ occurs on $\mathcal{B}$ and $\mathfrak{S}\omega_i\omega_j\tau_{[k]}$ iff*

[12] Note that it is often possible to produce shorter proofs or derivations by using some more 'intuitive' method instead.

$sw_iw_jt_k$ occurs on $\mathcal{B}$. $v(a) = [a]$, and $\langle [a_1], \dots, [a_n] \rangle \in v_{\omega_i \tau_{[j]}}(P)$ iff $Pa_1 \dots a_n, w_it_j$ is on $\mathcal{B}$, given that P is any n-place predicate other than identity. If $\neg Ra_m, w_it_j$ occurs on $\mathcal{B}$ and M is an n-place matrix with instantiations on the branch (where x_m is the first free variable in M and a_m is the constant in $M[a_1, \dots, a_n/\vec{x}]$ that replaces x_m), then $\langle [a_1], \dots, [a_n] \rangle \in v_{\omega_i \tau_{[j]}}(M)$ iff $M[a_1, \dots, a_n/\vec{x}], w_it_j$ occurs on $\mathcal{B}$. (Due to the identity rules this is well defined.) When we have $a = b, w_0t_0$, $b = c, w_0t_0$, etc. we choose one single object for all constants to denote.

If our tableau system includes neither $T - FC$, $T - PC$ nor $T - C$, $\rightleftharpoons$ is reduced to identity and $[i] = \{i\}$. Hence, in such systems, we may take T to be $\{\tau_i : t_i \text{ occurs on } \mathcal{B}\}$ and dispense with the equivalence classes.

Lemma 14 ***(Completeness Lemma)*** *Let $\mathcal{B}$ be an open branch in a complete tableau and let $\mathcal{M}$ be a model induced by $\mathcal{B}$. Then, for every formula A:*

(i) if A, w_it_j is on $\mathcal{B}$, then $\mathcal{M}, \omega_i, \tau_{[j]} \Vdash A$, and
(ii) if $\neg A, w_it_j$ is on $\mathcal{B}$, then $\mathcal{M}, \omega_i, \tau_{[j]} \nVdash A$.

Proof. The proof is by induction on the complexity of A.

(i) Atomic formulas. $Pa_1 \dots a_n, w_it_j$ is on $\mathcal{B} \Rightarrow \langle [a_1], \dots, [a_n] \rangle \in v_{\omega_i \tau_{[j]}}(P) \Rightarrow \langle v(a_1), \dots, v(a_n) \rangle \in v_{\omega_i \tau_{[j]}}(P) \Rightarrow \mathcal{M}, \omega_i, \tau_{[j]} \Vdash Pa_1 \dots a_n$.

$a = b, w_it_j$ is on $\mathcal{B} \Rightarrow a \sim b$ $(T - N =) \Rightarrow [a] = [b] \Rightarrow v(a) = v(b) \Rightarrow \mathcal{M}, \omega_i, \tau_{[j]} \Vdash a = b$.

Suppose that M is a matrix where x_m is the first free variable and a_m is the constant in $M[a_1, \dots, a_n/\vec{x}]$ that replaces x_m and that $\mathcal{M}, \omega_i, \tau_{[j]} \nVdash Ra_m$. Then: $M[a_1, \dots, a_n/\vec{x}], w_it_j$ occurs on $\mathcal{B} \Rightarrow \langle [a_1], \dots, [a_n] \rangle \in v_{\omega_i \tau_{[j]}}(M) \Rightarrow \langle v(a_1), \dots, v(a_n) \rangle \in v_{\omega_i \tau_{[j]}}(M) \Rightarrow \mathcal{M}, \omega_i, \tau_{[j]} \Vdash M[a_1, \dots, a_n/\vec{x}]$.

Other truth-functional connectives and modal, temporal and deontic operators. Straightforward.

Boulesic operators. ($\mathcal{W}$). Suppose $\mathcal{W}_cD, w_it_k$ is on $\mathcal{B}$. Moreover, suppose that Rc, w_it_k is not on $\mathcal{B}$. Then $\neg Rc, w_it_k$ is on $\mathcal{B}$ [by $CUTR$]. Hence, $\mathcal{W}_cD$ is true in ω_i at $\tau_{[k]}$ by definition and previous steps. Suppose Rc, w_it_k is on $\mathcal{B}$. Then since the branch is complete, the $\mathcal{W}$-rule has been applied and for every w_j such that $Acw_iw_jt_k$ is on $\mathcal{B}$, D, w_jt_k is on $\mathcal{B}$. By the induction hypothesis, D is true in every ω_j at $\tau_{[k]}$ such that $\mathfrak{A}v(c)\omega_i\omega_j\tau_{[k]}$. Since Rc, w_it_k is on $\mathcal{B}$, $v(c)$ is perfectly rational in ω_i at $\tau_{[k]}$. It follows that $\mathcal{W}_cD$ is true in ω_i at $\tau_{[k]}$, as required.

Other boulesic operators. Similar.

Quantifiers. (Σ). Suppose that $\Sigma xD, w_it_j$ is on the branch. Since the tableau is complete (Σ) has been applied. Accordingly, for some c, $D[c/x], w_it_j$ is on the branch. Hence, $\mathcal{M}, \omega_i, \tau_{[j]} \Vdash D[c/x]$, by (IH). For some $k_d \in \mathcal{L}(\mathcal{M})$, $v(c) = d$, and $v(k_d) = d$. Consequently, $\mathcal{M}, \omega_i, \tau_{[j]} \Vdash D[k_d/x]$, by the Denotation Lemma. It follows that $\mathcal{M}, \omega_i, \tau_{[j]} \Vdash \Sigma xD$.

The case for Π is similar.

(ii) Atomic formulas.

$\neg Pa_1 \dots a_n, w_it_j$ is on $\mathcal{B} \Rightarrow Pa_1 \dots a_n, w_it_j$ is not on $\mathcal{B}$ ($\mathcal{B}$ open) $\Rightarrow \langle [a_1], \dots, [a_n] \rangle \notin v_{\omega_i \tau_{[j]}}(P) \Rightarrow \langle v(a_1), \dots, v(a_n) \rangle \notin v_{\omega_i \tau_{[j]}}(P) \Rightarrow \mathcal{M}, \omega_i, \tau_{[j]} \nVdash Pa_1 \dots a_n$.

$\neg a = b, w_i t_j$ is on $\mathcal{B} \Rightarrow a = b, w_0 t_0$ is not on $\mathcal{B}$ ($\mathcal{B}$ open) $\Rightarrow$ it is not the case that $a \sim b \Rightarrow [a] \neq [b] \Rightarrow v(a) \neq v(b) \Rightarrow \mathcal{M}, \omega_i, \tau_{[j]} \nVdash a = b$.

Suppose that M is a matrix where x_m is the first free variable and a_m is the constant in $M[a_1, \ldots, a_n / \vec{x}]$ that replaces x_m and that $\mathcal{M}, \omega_i, \tau_{[j]} \nVdash Ra_m$. Then: $\neg M[a_1, \ldots, a_n / \vec{x}], w_i t_j$ occurs on $\mathcal{B} \Rightarrow M[a_1, \ldots, a_n / \vec{x}], w_i t_j$ is not on $\mathcal{B}$ ($\mathcal{B}$ open) $\Rightarrow \langle [a_1], \ldots, [a_n] \rangle \notin v_{\omega_i \tau_{[j]}}(M) \Rightarrow \langle v(a_1), \ldots, v(a_n) \rangle \notin v_{\omega_i \tau_{[j]}}(M) \Rightarrow \mathcal{M}, \omega_i, \tau_{[j]} \nVdash M[a_1, \ldots, a_n / \vec{x}]$.

Other truth-functional connectives and modal, temporal and deontic operators.

Straightforward.

Boulesic operators. ($\neg\mathcal{W}$). Suppose $\neg\mathcal{W}_c D, w_i t_k$ is on $\mathcal{B}$. Furthermore, suppose that $Rc, w_i t_k$ is not on $\mathcal{B}$. Then $\neg Rc, w_i t_k$ is on $\mathcal{B}$ [by $CUTR$]. Hence, $\mathcal{W}_c D$ is false in ω_i at $\tau_{[k]}$ by definition and previous steps. Suppose $Rc, w_i t_k$ is on $\mathcal{B}$. Then the $\neg\mathcal{W}$-rule has been applied to $\neg\mathcal{W}_c D, w_i t_k$ and we have $\mathcal{A}_c \neg D, w_i t_k$ on $\mathcal{B}$. For the branch is complete. Then the $\mathcal{A}$-rule has been applied to $\mathcal{A}_c \neg D, w_i t_k$, since the branch is complete. Hence, for some new w_j, $Acw_i w_j t_k$ and $\neg D, w_j t_k$ occur on $\mathcal{B}$. By the induction hypothesis, $\mathfrak{A} v(c) \omega_i \omega_j \tau_{[k]}$, and D is false in ω_j at $\tau_{[k]}$. Since $Rc, w_i t_k$ is on $\mathcal{B}$, $v(c)$ is perfectly rational in ω_i at $\tau_{[k]}$. Consequently, $\mathcal{W}_c D$ is false in ω_i at $\tau_{[k]}$, as required.

Other boulesic operators. Similar.

Quantifiers. ($\neg\Sigma$). Suppose that $\neg\Sigma x D, w_i t_j$ is on the branch. Since the tableau is complete ($\neg\Sigma$) has been applied. So, $\Pi x \neg D, w_i t_j$ is on the branch. Again, since the tableau is complete (Π) has been applied. Thus, for all $c \in C$, $\neg D[c/x], w_i t_j$ is on the branch. Consequently, $\mathcal{M}, \omega_i, \tau_{[j]} \nVdash D[c/x]$ for all $c \in C$ [by the induction hypothesis]. If $k_d \in \mathcal{L}(\mathcal{M})$, then for some $c \in C$, $v(c) = v(k_d)$. By the Denotation Lemma, for all $k_d \in \mathcal{L}(\mathcal{M})$, $\mathcal{M}, \omega_i, \tau_{[j]} \nVdash D[k_d/x]$. Consequently, $\mathcal{M}, \omega_i, \tau_{[j]} \nVdash \Sigma x D$.

($\neg\Pi$). Straightforward. ■

Theorem 15 ***(Completeness Theorem)*** *Every system in this paper is complete with respect to its semantics.*

Proof. First, I will show that the theorem holds for our weakest system $\mathcal{Q}$. Then, I will extend the theorem to all stronger systems. Let $\boldsymbol{M}$ be the class of models that corresponds to $\mathcal{Q}$.

Suppose that B is not derivable from Γ in $\mathcal{Q}$. Then it is not the case that there is a closed $\mathcal{Q}$-tableau that starts with $A, w_0 t_0$ for every A in Γ and $\neg B, w_0 t_0$. Let t be a complete $\mathcal{Q}$-tableau whose first nodes comprises $A, w_0 t_0$ for every A in Γ and $\neg B, w_0 t_0$. Obviously, t is not closed; it is open. It follows that there is at least one open branch in t. Let $\mathcal{B}$ be an open branch in t. According to the model induced by $\mathcal{B}$, all the premises in Γ are true and B false in ω_0 at $\tau_{[0]}$. Hence, it is not the case that B follows from Γ in $\boldsymbol{M}$. Consequently, if B follows from Γ in $\boldsymbol{M}$, then B is derivable from Γ in $\mathcal{Q}$.

I will now show that all extensions of $\mathcal{Q}$ are complete with respect to their semantics. To establish this we have to verify that the model induced by the open branch $\mathcal{B}$ is of the right kind in every case. First, we must go through every single semantic condition and prove that the induced model is of the right kind. Then we combine our proofs. The following steps illustrate the method:

$C - b5$. Suppose that $\mathfrak{A} v(c) \omega_i \omega_j \tau_{[l]}$ and $\mathfrak{A} v(c) \omega_i \omega_k \tau_{[l]}$. Then, both $Acw_i w_j t_l$ and $Acw_i w_k t_l$ occur on $\mathcal{B}$ [by the definition of an induced model]. Since $\mathcal{B}$ is complete,

$(T-b5)$ has been applied and $Acw_jw_kt_l$ occurs on $\mathcal{B}$. It follows that $\mathfrak{A}v(c)\omega_j\omega_k\tau_{[l]}$, as required [by the definition of an induced model].

$C-t4$. Suppose that $\tau_{[i]} < \tau_{[j]}$ and $\tau_{[j]} < \tau_{[k]}$. Then $t_i < t_j$ and $t_j < t_k$ occur on $\mathcal{B}$ [by the definition of an induced model]. Since $\mathcal{B}$ is complete $(T-t4)$ has been applied. Hence, $t_i < t_k$ occurs on $\mathcal{B}$. It follows that $\tau_{[i]} < \tau_{[k]}$, as required [by the definition of an induced model].

$C-ad5$. Assume that $\mathfrak{R}\omega_i\omega_j\tau_{[l]}$ and $\mathfrak{S}\omega_i\omega_k\tau_{[l]}$. Then, both $rw_iw_jt_l$ and $sw_iw_kt_l$ occur on $\mathcal{B}$ [by the definition of an induced model]. Since $\mathcal{B}$ is complete, $(T-ad5)$ has been applied and $sw_jw_kt_l$ occurs on $\mathcal{B}$. Hence, $\mathfrak{S}\omega_j\omega_k\tau_{[l]}$, as required [by the definition of an induced model].

$C-\Box\mathcal{W}$. Suppose that $\mathfrak{A}v(c)\omega_i\omega_j\tau_{[k]}$. Then $Acw_iw_jt_k$ occurs on $\mathcal{B}$ [by the definition of an induced model]. Since $\mathcal{B}$ is complete, $(T-\Box\mathcal{W})$ has been applied and $rw_iw_jt_k$ occurs on $\mathcal{B}$. Consequently, $\mathfrak{R}\omega_i\omega_j\tau_{[k]}$, as required [by the definition of an induced model].

$C-ASP$. Suppose that $\mathfrak{R}\omega_i\omega_j\tau_{[l]}$ and $\tau_{[k]} < \tau_{[l]}$. Then $rw_iw_jt_l$ and $t_k < t_l$ occur on $\mathcal{B}$ [by the definition of an induced model]. Since $\mathcal{B}$ is complete $(T-ASP)$ has been applied. Hence, $rw_iw_jt_k$ occurs on $\mathcal{B}$. It follows that $\mathfrak{R}\omega_i\omega_j\tau_{[k]}$, as required [by the definition of an induced model].

$C-\mathbf{O}\mathbb{G}\mathbf{O}\Diamond$. Suppose that $\mathfrak{S}\omega_i\omega_j\tau_{[l]}$ and $\tau_{[l]} < \tau_{[m]}$. Then $sw_iw_jt_l$ and $t_l < t_m$ occur on $\mathcal{B}$ [by the definition of an induced model]. Since $\mathcal{B}$ is complete $(T-\mathbf{O}\mathbb{G}\mathbf{O}\Diamond)$ has been applied. Hence, for some w_k, $rw_jw_kt_m$ and $sw_jw_kt_m$ are on $\mathcal{B}$. Accordingly, for some ω_k, $\mathfrak{R}\omega_j\omega_k\tau_{[m]}$ and $\mathfrak{S}\omega_j\omega_k\tau_{[m]}$, as required [by the definition of an induced model].

$C-\mathcal{W}\mathbb{G}\Box\mathcal{W}$. Suppose that $\mathfrak{A}v(c)\omega_i\omega_j\tau_{[l]}$, $\tau_{[l]} < \tau_{[m]}$ and $\mathfrak{A}v(c)\omega_j\omega_k\tau_{[m]}$. Then $Acw_iw_jt_l$, $t_l < t_m$ and $Acw_jw_kt_m$ occur on $\mathcal{B}$ [by the definition of an induced model]. Since $\mathcal{B}$ is complete $(T-\mathcal{W}\mathbb{G}\Box\mathcal{W})$ has been applied. Accordingly, $rw_jw_kt_m$ occurs on $\mathcal{B}$. Consequently, $\mathfrak{R}\omega_j\omega_k\tau_{[m]}$, as required [by the definition of an induced model]. ■

7 Examples

In this section, I will consider one example of a valid sentence, one example of an invalid sentence and one example of a valid argument. I will show how one can use semantic tableaux to construct proofs and derivations and to read off countermodels from open and complete trees. All the examples in this section were mentioned in the introduction.

7.1 Example 1: A valid sentence

I will now show that the following sentence is a theorem in every system that includes $T-\Box\mathcal{W}$[13]:

E1. It is (absolutely) necessary that if a perfectly rational individual x wants it to be the case that A sometime in the future and it is (historically) necessary that it is always

[13] In a strict sense **E1** is not a sentence but a schema. When we say that this sentence is a theorem we mean that every instance of it is a theorem.

going to be the case that if A then B, then x wants it to be the case that B sometime in the future.

This sentence can be symbolised in the following way in our systems: $\mathbb{U}\Pi x(Rx \to ((\mathcal{W}_x\mathbb{F}A \wedge \Box\mathbb{G}(A \to B)) \to \mathcal{W}_x\mathbb{F}B))$. A proof of a sentence A in a system S is a closed S-tableau that begins with $\neg A, w_0t_0$. If there is a proof of A in S, A is a theorem in S (see Definition 6). Consequently, to prove that $\mathbb{U}\Pi x(Rx \to ((\mathcal{W}_x\mathbb{F}A \wedge \Box\mathbb{G}(A \to B)) \to \mathcal{W}_x\mathbb{F}B))$ is a theorem in every $T - \Box\mathcal{W}$-system we construct a closed $T - \Box\mathcal{W}$-tableau for the negation of this sentence. More precisely, we construct a closed $T - \Box\mathcal{W}$-tableau whose root consists of (1) below. Here is the proof (MP is an abbreviation of the derived rule Modus Ponens):

$$(1)\ \neg\mathbb{U}\Pi x(Rx \to ((\mathcal{W}_x\mathbb{F}A \wedge \Box\mathbb{G}(A \to B)) \to \mathcal{W}_x\mathbb{F}B)), w_0t_0$$
$$(2)\ \mathbb{M}\neg\Pi x(Rx \to ((\mathcal{W}_x\mathbb{F}A \wedge \Box\mathbb{G}(A \to B)) \to \mathcal{W}_x\mathbb{F}B)), w_0t_0\ [1, \neg\mathbb{U}]$$
$$(3)\ \neg\Pi x(Rx \to ((\mathcal{W}_x\mathbb{F}A \wedge \Box\mathbb{G}(A \to B)) \to \mathcal{W}_x\mathbb{F}B)), w_1t_1\ [2, \mathbb{M}]$$
$$(4)\ \Sigma x\neg(Rx \to ((\mathcal{W}_x\mathbb{F}A \wedge \Box\mathbb{G}(A \to B)) \to \mathcal{W}_x\mathbb{F}B)), w_1t_1\ [3, \neg\Pi]$$
$$(5)\ \neg(Rc \to ((\mathcal{W}_c\mathbb{F}A \wedge \Box\mathbb{G}(A \to B)) \to \mathcal{W}_c\mathbb{F}B)), w_1t_1\ [4, \Sigma]$$
$$(6)\ Rc, w_1t_1\ [5, \neg\to]$$
$$(7)\ \neg((\mathcal{W}_c\mathbb{F}A \wedge \Box\mathbb{G}(A \to B)) \to \mathcal{W}_c\mathbb{F}B), w_1t_1\ [5, \neg\to]$$
$$(8)\ \mathcal{W}_c\mathbb{F}A \wedge \Box\mathbb{G}(A \to B), w_1t_1\ [7, \neg\to]$$
$$(9)\ \neg\mathcal{W}_c\mathbb{F}B, w_1t_1\ [7, \neg\to]$$
$$(10)\ \mathcal{W}_c\mathbb{F}A, w_1t_1\ [8, \wedge]$$
$$(11)\ \Box\mathbb{G}(A \to B), w_1t_1\ [8, \wedge]$$
$$(12)\ \mathcal{A}_c\neg\mathbb{F}B, w_1t_1\ [6, 9, \neg\mathcal{W}]$$
$$(13)\ Acw_1w_2t_1\ [6, 12, \mathcal{A}]$$
$$(14)\ \neg\mathbb{F}B, w_2t_1\ [6, 12, \mathcal{A}]$$
$$(15)\ \mathbb{G}\neg B, w_2t_1\ [14, \neg\mathbb{F}]$$
$$(16)\ \mathbb{F}A, w_2t_1\ [6, 10, 13, \mathcal{W}]$$
$$(17)\ rw_1w_2t_1\ [13, T - \Box\mathcal{W}]$$
$$(18)\ \mathbb{G}(A \to B), w_2t_1\ [11, 17, \Box]$$
$$(19)\ t_1 < t_2\ [16, \mathbb{F}]$$
$$(20)\ A, w_2t_2\ [16, \mathbb{F}]$$
$$(21)\ A \to B, w_2t_2\ [18, 19, \mathbb{G}]$$
$$(22)\ \neg B, w_2t_2\ [15, 19, \mathbb{G}]$$
$$(23)\ B, w_2t_2\ [20, 21, MP]$$
$$(24)\ *\ [22, 23]$$

The smallest system that includes $T - \Box\mathcal{W}$ is valid with respect to the class of all $C - \Box\mathcal{W}$-models (see Section 6). It follows that $\mathbb{U}\Pi x(Rx \to ((\mathcal{W}_x\mathbb{F}A \wedge \Box\mathbb{G}(A \to B)) \to \mathcal{W}_x\mathbb{F}B))$ is valid in the class of all models that satisfy $C - \Box\mathcal{W}$.

7.2 Example 2: An invalid sentence and a countermodel

We have seen that $\mathbb{U}\Pi x(Rx \to ((\mathcal{W}_x\mathbb{F}A \wedge \Box\mathbb{G}(A \to B)) \to \mathcal{W}_x\mathbb{F}B))$ is valid in the class of all $C - \Box\mathcal{W}$-models. However, the sentence is not valid in the class of all models. Nor can we show that the following proposition is valid: it is (absolutely) necessary that if some individual x wants it to be the case that A sometime in the future and it is (historically) necessary that it is always going to be the case that if A then B, then x wants it to be the case that B sometime in the future. In fact, this sentence cannot be proved in any system in this paper. In the proof above (Section 7.1), it is essential that we are quantifying over perfectly rational individuals. All of this is intuitively plausible.

I will now show that the following sentence is not valid in the class of all models:

E2. If an individual x wants it to be the case that x sometime in the future will be a citizen of Great Britain and it is (historically) necessary that it is always going to be the case that if x is a citizen of Great Britain then x is a citizen of Europe, then x wants it to be the case that x sometime in the future will be a citizen of Europe.

E2 can be symbolised in the following way: $\Pi x((\mathcal{W}_x \mathbb{F} Gx \wedge \Box \mathbb{G}(Gx \to Ux)) \to \mathcal{W}_x \mathbb{F} Ux)$, where '$Gx$' reads as '$x$ is a citizen of Great Britain' and 'Ux' reads as 'x is a citizen of Europe'. To show that a sentence A is not valid we construct an open complete tableau for the negation of this sentence. More precisely, we construct an open semantic tableau that begins with $\neg A, w_0 t_0$. Then we use an open branch in the tree to read off a countermodel. Here is our tableau:

$$
\begin{array}{c}
(1)\ \neg\Pi x((\mathcal{W}_x \mathbb{F} Gx \wedge \Box \mathbb{G}(Gx \to Ux)) \to \mathcal{W}_x \mathbb{F} Ux), w_0 t_0 \\
(2)\ \Sigma x \neg((\mathcal{W}_x \mathbb{F} Gx \wedge \Box \mathbb{G}(Gx \to Ux)) \to \mathcal{W}_x \mathbb{F} Ux), w_0 t_0\ [1, \neg\Pi] \\
(3)\ \neg((\mathcal{W}_c \mathbb{F} Gc \wedge \Box \mathbb{G}(Gc \to Uc)) \to \mathcal{W}_c \mathbb{F} Uc), w_0 t_0\ [2, \Sigma] \\
(4)\ \mathcal{W}_c \mathbb{F} Gc \wedge \Box \mathbb{G}(Gc \to Uc), w_0 t_0\ [3, \neg \to] \\
(5)\ \neg \mathcal{W}_c \mathbb{F} Uc, w_0 t_0\ [3, \neg \to] \\
(6)\ \mathcal{W}_c \mathbb{F} Gc, w_0 t_0\ [4, \wedge] \\
(7)\ \Box \mathbb{G}(Gc \to Uc), w_0 t_0\ [4, \wedge] \\
\swarrow \qquad \qquad \searrow \\
\begin{array}{ll}
(8)\ Rc, w_0 t_0 & (9)\ \neg Rc, w_0 t_0\ [CUTR] \\
 & (10)\ c = c, w_0 t_0\ [T - R =]
\end{array}
\end{array}
$$

The left branch in this tree can be developed further. However, at this stage we cannot apply any more rules to the right branch, which is open (and complete). Hence, the whole tableau is open (and complete). So, $\Pi x((\mathcal{W}_x \mathbb{F} Gx \wedge \Box \mathbb{G}(Gx \to Ux)) \to \mathcal{W}_x \mathbb{F} Ux)$ is not a theorem in our weakest system $\mathcal{Q}$.[14] Accordingly, the formula is invalid in the class of all models (by the completeness results in Section 6).

Let us now verify this claim. Since the right branch in the tree is open and complete, we can use it to read off a countermodel $\mathcal{M}$. The matrix of $\mathcal{W}_c \mathbb{F} Uc$ is $\mathcal{W}_{x_1} \mathbb{F} Ux_2$, and the matrix of $\mathcal{W}_c \mathbb{F} Gc$ is $\mathcal{W}_{x_1} \mathbb{F} Gx_2$. $W = \{\omega_0\}$, $T = \{\tau_0\}$, $D = \{[c]\}$, $v(c) = [c]$, and the extensions of G and U are empty in ω_0 at τ_0. $<$, $\mathfrak{R}$, $\mathfrak{A}$ and $\mathfrak{S}$ are empty. $v_{\omega_0 \tau_0}(\mathcal{W}_{x_1} \mathbb{F} Ux_2)$ is the extension of $\mathcal{W}_{x_1} \mathbb{F} Ux_2$ in ω_0 at τ_0, and $v_{\omega_0 \tau_0}(\mathcal{W}_{x_1} \mathbb{F} Gx_2)$ is the extension of $\mathcal{W}_{x_1} \mathbb{F} Gx_2$ in ω_0 at τ_0. If $\neg Ra_m, w_i t_j$ occurs on the branch $\mathcal{B}$ and M is an n-place matrix with instantiations on the branch (where x_m is the first free variable in M and a_m is the constant in $M[a_1, \ldots, a_n / x_1, \ldots, x_n]$ that replaces x_m),

[14] In fact, the sentence is not a theorem in *any* system in this paper. However, it is left to the reader to verify this.

then $\langle[a_1],\ldots,[a_n]\rangle$ is an element of $v_{\omega_i\tau_j}(M)$ iff $M[a_1,\ldots,a_n/x_1,\ldots,x_n],w_it_j$ occurs on $\mathcal{B}$. $\neg Rc, w_0t_0$ occurs on the branch, while $\mathcal{W}_{x_1}\mathbb{F}Ux_2[c,c/x_1,x_2],w_0t_0$ (that is, $\mathcal{W}_c\mathbb{F}Uc, w_0t_0$) does not occur on the branch. x_1 is the first free variable in $\mathcal{W}_{x_1}\mathbb{F}Ux_2$ and c is the constant in $\mathcal{W}_{x_1}\mathbb{F}Ux_2[c,c/x_1,x_2]$ that replaces x_1. So, $\langle[c],[c]\rangle$ is not an element in $v_{\omega_0\tau_0}(\mathcal{W}_{x_1}\mathbb{F}Ux_2)$ ($v_{\omega_0\tau_0}(\mathcal{W}_{x_1}\mathbb{F}Ux_2)$ is empty). Since $\neg Rc, w_0t_0$ occurs on $\mathcal{B}$, Rc is false in ω_0 at τ_0. If $\mathcal{M},\omega_0,\tau_0 \nVdash Rc$, then $\mathcal{M},\omega_0,\tau_0 \Vdash \mathcal{W}_{x_1}\mathbb{F}Ux_2[c,c/x_1,x_2]$ iff $\langle v(c),v(c)\rangle$ is in $v_{\omega_0\tau_0}(\mathcal{W}_{x_1}\mathbb{F}Ux_2)$. Hence, $\mathcal{M},\omega_0,\tau_0 \Vdash \mathcal{W}_{x_1}\mathbb{F}Ux_2[c,c/x_1,x_2]$ iff $\langle v(c),v(c)\rangle$ is in $v_{\omega_0\tau_0}(\mathcal{W}_{x_1}\mathbb{F}Ux_2)$. $\langle v(c),v(c)\rangle$ is not in $v_{\omega_0\tau_0}(\mathcal{W}_{x_1}\mathbb{F}Ux_2)$. Therefore, it is not the case that $\mathcal{M},\omega_0,\tau_0 \Vdash \mathcal{W}_{x_1}\mathbb{F}Ux_2[c,c/x_1,x_2]$. $\mathcal{W}_{x_1}\mathbb{F}Ux_2[c,c/x_1,x_2] = \mathcal{W}_c\mathbb{F}Uc$. It follows that it is not the case that $\mathcal{M},\omega_0,\tau_0 \Vdash \mathcal{W}_c\mathbb{F}Uc$, that is, $\mathcal{W}_c\mathbb{F}Uc$ is false in ω_0 at τ_0.

$\mathcal{W}_{x_1}\mathbb{F}Gx_2[c,c/x_1,x_2],w_0t_0$ (that is, $\mathcal{W}_c\mathbb{F}Gc, w_0t_0$) occurs on the branch. x_1 is the first free variable in $\mathcal{W}_{x_1}\mathbb{F}Gx_2$ and c is the constant in $\mathcal{W}_{x_1}\mathbb{F}Gx_2[c,c/x_1,x_2]$ that replaces x_1. Accordingly, $\langle[c],[c]\rangle$ is an element in $v_{\omega_0\tau_0}(\mathcal{W}_{x_1}\mathbb{F}Gx_2)$. If $\mathcal{M},\omega_0,\tau_0 \nVdash Rc$, then $\mathcal{M},\omega_0,\tau_0 \Vdash \mathcal{W}_{x_1}\mathbb{F}Gx_2[c,c/x_1,x_2]$ iff $\langle v(c),v(c)\rangle$ is in $v_{\omega_0\tau_0}(\mathcal{W}_{x_1}\mathbb{F}Gx_2)$. Hence, $\mathcal{M},\omega_0,\tau_0 \Vdash \mathcal{W}_{x_1}\mathbb{F}Gx_2[c,c/x_1,x_2]$ iff $\langle v(c),v(c)\rangle$ is in $v_{\omega_0\tau_0}(\mathcal{W}_{x_1}\mathbb{F}Gx_2)$. $\langle v(c),v(c)\rangle$ is in $v_{\omega_0\tau_0}(\mathcal{W}_{x_1}\mathbb{F}Gx_2)$. Therefore, $\mathcal{M},\omega_0,\tau_0 \Vdash \mathcal{W}_{x_1}\mathbb{F}Gx_2[c,c/x_1,x_2]$. It follows that $\mathcal{M},\omega_0,\tau_0 \Vdash \mathcal{W}_c\mathbb{F}Gc$, for $\mathcal{W}_{x_1}\mathbb{F}Gx_2[c,c/x_1,x_2] = \mathcal{W}_c\mathbb{F}Gc$. In other words, $\mathcal{W}_c\mathbb{F}Gc$ is true in ω_0 at τ_0.

Since $\mathfrak{R}$ is empty, $\Box\mathbb{G}(Gc \to Uc)$ is vacuously true in ω_0 at τ_0. Hence, $\mathcal{W}_c\mathbb{F}Gc \wedge \Box\mathbb{G}(Gc \to Uc)$ is true in ω_0 at τ_0. Accordingly, $(\mathcal{W}_c\mathbb{F}Gc \wedge \Box\mathbb{G}(Gc \to Uc)) \to \mathcal{W}_c\mathbb{F}Uc$ is false in ω_0 at τ_0. Since $[c]$ is an object in the domain, it follows that $\Pi x((\mathcal{W}_x\mathbb{F}Gx \wedge \Box\mathbb{G}(Gx \to Ux)) \to \mathcal{W}_x\mathbb{F}Ux)$ is false in ω_0 at τ_0. In conclusion, $\mathbb{U}\Pi x((\mathcal{W}_x\mathbb{F}Gx \wedge \Box\mathbb{G}(Gx \to Ux)) \to \mathcal{W}_x\mathbb{F}Ux)$ is not valid in the class of all models. Q.E.D.

7.3 Example 3: A valid argument

In this section, I will show that the following argument is valid in the class of all models:

E3. $P1$. $\mathbb{U}(\neg\Sigma x(Rx \wedge \mathcal{A}_x\mathbb{F}Pu) \to \neg\mathbf{P}\mathbb{F}Pu)$ ('u' refers to you and 'Px' says that 'x rapes someone'[15]). It is (absolutely) necessary that if no perfectly rational individual accepts that you will rape someone in the future, then it is not permitted that you will rape someone in the future.

$P2$. $\Pi x(Rx \to \mathcal{W}_x\mathbb{G}\neg Pu)$. Everyone who is perfectly rational wants it to be the case that it is always going to be the case that you do not rape someone.

Hence,

C. $\mathbf{O}\mathbb{G}\neg Pu$. It ought to be the case that it is always going to be the case that you do not rape someone.

To show that the conclusion (C) is derivable from the premises ($P1$) and ($P2$) in every system in this paper, we construct a closed tableau that starts with $P1, w_0t_0$, $P2, w_0t_0$ and $\neg C, w_0t_0$. Since we do not use any special tableau rules in the tree, the conclusion

[15] 'x rapes someone' can also be symbolised in the following way: $\Sigma yPxy$, where 'Pxy' says that x rapes y, but we do not need to use this more 'sophisticated' analysis to prove that the argument is valid. So, we will stick to the monadic predicate P.

is derivable from the premises in our weakest system. Consequently, it is derivable in every other system too. By the soundness results in Section 6, it follows that the argument is valid in the class of all models. This establishes the desired result.

Here is the tableau that proves that the conclusion is derivable from the premises in every system:

$$(1)\ \mathbb{U}(\neg\Sigma x(Rx \wedge \mathcal{A}_x\mathbb{F}Pu) \to \neg\mathbf{P}\mathbb{F}Pu), w_0t_0$$
$$(2)\ \Pi x(Rx \to \mathcal{W}_x\mathbb{G}\neg Pu), w_0t_0$$
$$(3)\ \neg\mathbf{O}\mathbb{G}\neg Pu, w_0t_0$$
$$(4)\ \mathbf{P}\neg\mathbb{G}\neg Pu, w_0t_0\ [3, \neg\mathbf{O}]$$
$$(5)\ sw_0w_1t_0\ [4, \mathbf{P}]$$
$$(6)\ \neg\mathbb{G}\neg Pu, w_1t_0\ [4, \mathbf{P}]$$
$$(7)\ \mathbb{F}\neg\neg Pu, w_1t_0\ [6, \neg\mathbb{G}]$$
$$(8)\ t_0 < t_1\ [7, \mathbb{F}]$$
$$(9)\ \neg\neg Pu, w_1t_1\ [7, \mathbb{F}]$$
$$(10)\ \neg\Sigma x(Rx \wedge \mathcal{A}_x\mathbb{F}Pu) \to \neg\mathbf{P}\mathbb{F}Pu, w_0t_0\ [1, \mathbb{U}]$$

↙	↘
$(11)\ \neg\neg\Sigma x(Rx \wedge \mathcal{A}_x\mathbb{F}Pu), w_0t_0\ [10, \to]$	$(12)\ \neg\mathbf{P}\mathbb{F}Pu, w_0t_0\ [10, \to]$
$(13)\ \Sigma x(Rx \wedge \mathcal{A}_x\mathbb{F}Pu), w_0t_0\ [11, \neg\neg]$	$(14)\ \mathbf{O}\neg\mathbb{F}Pu, w_0t_0\ [12, \neg\mathbf{P}]$
$(15)\ Rc \wedge \mathcal{A}_c\mathbb{F}Pu, w_0t_0\ [13, \Sigma]$	$(16)\ \neg\mathbb{F}Pu, w_1t_0\ [5, 14, \mathbf{O}]$
$(17)\ Rc, w_0t_0\ [15, \wedge]$	$(18)\ \mathbb{G}\neg Pu, w_1t_0\ [16, \neg\mathbb{F}]$
$(19)\ \mathcal{A}_c\mathbb{F}Pu, w_0t_0\ [15, \wedge]$	$(20)\ \neg Pu, w_1t_1\ [8, 18, \mathbb{G}]$
$(21)\ Rc \to \mathcal{W}_c\mathbb{G}\neg Pu, w_0t_0\ [2, \Pi]$	$(22)\ \ast\ [9, 20]$
$(23)\ \mathcal{W}_c\mathbb{G}\neg Pu, w_0t_0\ [17, 21, MP]$	
$(24)\ Acw_0w_2t_0\ [17, 19, \mathcal{A}]$	
$(25)\ \mathbb{F}Pu, w_2t_0\ [17, 19, \mathcal{A}]$	
$(26)\ \mathbb{G}\neg Pu, w_2t_0\ [17, 23, 24, \mathcal{W}]$	
$(27)\ t_0 < t_2\ [25, \mathbb{F}]$	
$(28)\ Pu, w_2t_2\ [25, \mathbb{F}]$	
$(29)\ \neg Pu, w_2t_2\ [26, 27, \mathbb{G}]$	
$(30)\ \ast\ [28, 29]$	

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