Axioms for Hansson's Dyadic Deontic Logics

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Abstract

This paper presents axiomatic systems equivalent to Bengt Hansson's semantically defined dyadic deontic logics, DSDL1, DSDL2 and DSDL3. Each axiomatic system is demonstrated to be sound and complete with respect to the particular classes of models Hansson defined, and in that way to be equivalent to his logics. I also include another similar member of the family I call DSDL2.5 and provide an axiomatic system for it. These systems are further found to be decidable, and, although DSDL3 is compact, the three weaker ones are shown not to be.

1 Introduction

2019 marks the 50th anniversary of the publication of Bengt Hansson's seminal paper "An Analysis of Some Deontic Logics" [5]. While there were precursors and other contemporary works developing similar concepts, Hansson's paper, perhaps more than any other, set the stage for research into dyadic deontic logics, while it also stimulated the use of semantical methods for the analysis of normative language. In that paper Hansson proposed a way to model statements of conditional obligation, or obligation under circumstances, with a pattern that, broadly speaking, has since become quite standard, even commonplace. From those constructions he then defined three systems of dyadic deontic logic, DSDL1, DSDL2 and DSDL3, of increasing strength. These were defined entirely semantically. Although Hansson presented various principles that are valid, or not, for each logic, he seemed little interested in axiomatic or proof-theoretic treatments of them. Later, Spohn [16] offered an axiomatization for the strongest, DSDL3. The other two have received far less attention, and the question of their axiomatization has remained open.[1]

In this paper I introduce axiomatizations for all three of Hansson's logics as he formulated them, and demonstrate their adequacy by proving them sound and complete with respect to the classes of models Hansson defined. I also include another member of the family, which I call DSDL2.5 since it is between DSDL2 and DSDL3. Regarding DSDL3, my version is a little different from Spohn's, though demonstrably equivalent to his. In the form I give, however, it is easy to see how, from an axiomatic point of view, this system is a natural extension of the others. Moreover, I prove a

[1] Some efforts to reconstruct DSDL1 and DSDL2 differ significantly from Hansson's account, e.g., in the language of the systems, the form of model applied to the language, even the rule of interpretation for dyadic deontic formulas; cf., e.g., Parent [10] or Åqvist [17, 18] with his systems **E** and **F**, and Parent's subsequent results for those systems, [12]. As a result, those efforts fall outside the framework of this paper.

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strong completeness result for it, and thus show DSDL3 to be compact, whereas Spohn provided only weak completeness.

The course of this paper is as follows: Section 2 presents the details of Hansson's semantics for dyadic deontic logic, and defines his systems DSDL1, DSDL2 and DSDL3, as well as DSDL2.5. This section also describes an alternative, though similar, way of modeling dyadic deontic statements. This will be useful for establishing later results. Section 3 defines the axiomatic systems we shall be studying; I call these DDL-a, DDL-b, DDL-c and DDL-d.

With Section 4 the work begins. Here we prove that DDL-d is equivalent to Hansson's DSDL3. The proof will be straightforward and its methods familiar. Though this system is the strongest of the family, we put it first because it is easiest to work with. The demonstrations for the others will, unfortunately but perhaps inevitably, be more difficult; that for DDL-b and DSDL2 is particularly challenging. Those demonstrations occupy Section 5, to establish the equivalence of DDL-a, DDL-b and DDL-c with DSDL1, DSDL2 and DSDL2.5, respectively. Following that, Section 6 presents some ancillary results that follow from those of the preceding sections, most notably that, in contrast to DSDL3, the weaker logics DSDL1, DSDL2 and DSDL2.5 are not compact, and then that all of these systems, including DSDL3, are decidable, subjects that Hansson did not address. Here too we examine a variation on the form of interpretation Hansson applied to dyadic deontic formulas, and find that the variation makes little difference to the systems determined by the semantic rules. Section 7 is merely a quick recap of what has been established.

Axiomatizing Hansson's logics in this way helps one better to understand and appreciate the commitments of these systems by identifying their fundamental principles. Nonetheless, throughout this paper I will only be concerned with the technical problems of the equivalence of the axiomatizations to Hansson's semantics. In particular, I do not address philosophical questions of the adequacy of the systems for problems in deontic logic or the analysis of normative discourse. Nor do I compare the virtues or vices of the different systems amongst themselves. I do not survey other work in dyadic deontic logic. Furthermore, the focus here is entirely on Hansson's DSDL logics. I do not discuss the various other topics and themes he raised in his paper.

2 Hansson's DSDL systems

Here I present Hansson's logics DSDL1, DSDL2, and DSDL3, essentially as he presented them, though in my own notation and style of doing things; I also add DSDL2.5. I specify the language of these logics, and the particular way Hansson modeled that language in terms of which he defined his systems. I then describe another similar form of model for the language, a form with, however, some noteworthy contrasts, which allow it to be more flexible than Hansson's own, and thus useful later on.

2.1 The austere language

Hansson formulated his systems of dyadic deontic logic only for an austere language that excludes formulas in which deontic modalities occur within the scope of other deontic modalities, and that also excludes mixed formulas, in which deontic formulas are truth-functionally combined with non-deontic formulas. Further, it contains no other modalities, such as for alethic necessity or possibility. This austerity is common in studies of deontic logic, though by no means universal.

Let us make that austere language more precise. It begins with the language of a 'base logic', $\mathcal{L}_{\mathrm{BL}}$, that we take now to be a language for the classical propositional calculus, with denumerably many atomic formulas p, q, r, $\ldots$, etc., and the usual connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, with the usual formation rules. $A \leftrightarrow B$ is defined as $(A \rightarrow B) \wedge (B \rightarrow A)$. $\top$ is any classical tautology and $\bot$ a classical contradiction in $\mathcal{L}_{\mathrm{BL}}$. I use '$A$', '$B$', '$C$', etc. as variables for formulas in this base language, $\mathcal{L}_{\mathrm{BL}}$.

The austere deontic language, $\mathcal{L}_{\mathrm{DL}^a}$, based on $\mathcal{L}_{\mathrm{BL}}$ contains the primitive dyadic deontic operator $O(-/-)$, such that $O(B/A)$ is well-formed and a member of $\mathcal{L}_{\mathrm{DL}^a}$ whenever, and only when, $A, B \in \mathcal{L}_{\mathrm{BL}}$. Informally, $O(B/A)$ may be read to say, If A then it ought to be that B, or B is obligatory, given circumstances A, cf. [5], p. 133. $\mathcal{L}_{\mathrm{DL}^a}$ also contains the Boolean connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, which we may take to be the same as for $\mathcal{L}_{\mathrm{BL}}$ but now also applied to formulas in $\mathcal{L}_{\mathrm{DL}^a}$, so that if $\alpha, \beta \in \mathcal{L}_{\mathrm{DL}^a}$, then, and only then, $\neg\alpha \in \mathcal{L}_{\mathrm{DL}^a}$, $\alpha \wedge \beta \in \mathcal{L}_{\mathrm{DL}^a}$, $\alpha \vee \beta \in \mathcal{L}_{\mathrm{DL}^a}$, and $\alpha \rightarrow \beta \in \mathcal{L}_{\mathrm{DL}^a}$, all as usual. That is the whole of $\mathcal{L}_{\mathrm{DL}^a}$. I use '$\alpha$', '$\beta$', '$\gamma$', etc. as variables for formulas of $\mathcal{L}_{\mathrm{DL}^a}$. $P(B/A)$, for conditional permission, is defined as $\neg O(\neg B/A)$.

For $\mathcal{L}_{\mathrm{BL}}$ to contain denumerably many atomic formulas is significant. If the language had a finite vocabulary, our results would be easier to obtain, though they would also be somewhat different. I do not pursue that difference here, however.

2.2 Hansson's models

To interpret formulas $A \in \mathcal{L}_{\mathrm{BL}}$, Hansson drew on standard classical logic. In effect, he identified possible worlds with familiar valuation functions. Let V be the set of all classical valuations for $\mathcal{L}_{\mathrm{BL}}$. I.e., for each $\varphi \in V$, φ is a function defined for every atomic formula $p \in \mathcal{L}_{\mathrm{BL}}$, such that $\varphi(p) \in \{1, 0\}$, and $\varphi(p) = 1$ iff $\varphi(p) \neq 0$. I use 'φ', with or without decoration by subscripts or superscripts, as variables over V.

The full interpretation function $|A|$ for formulas $A \in \mathcal{L}_{\mathrm{BL}}$ is specified in the usual way, to give the set of valuations under which A would be true:

$$
\begin{aligned}
&|p| = \{\varphi \in V : \varphi(p) = 1\},\\
&|\neg A| = V - |A|,\\
&|A \wedge B| = |A| \cap |B|,\\
&|A \vee B| = |A| \cup |B|,\\
&|A \rightarrow B| = (V - |A|) \cup |B|.
\end{aligned}
$$

We may presuppose, usually without remark, familiar results from classical logic, e.g., that $\vdash A \rightarrow B$ iff $|A| \subseteq |B|$ and $\vdash A \leftrightarrow B$ iff $|A| = |B|$, where $\vdash$ represents derivability within the classical propositional calculus. Often I will say a formula $A \in \mathcal{L}_{\mathrm{BL}}$ is 'BL-consistent', or simply 'consistent', and mean either that $|A| \neq \varnothing$, or that $\nvdash \neg A$. By the soundness and completeness of the calculus, the two come to the same thing. Similarly for sets of formulas $\mathbb{C} \subseteq \mathcal{L}_{\mathrm{BL}}$, $\mathbb{C}$ is consistent iff there is a $\varphi \in V$ such that $\varphi \in |C|$ for every $C \in \mathbb{C}$ iff $\mathbb{C} \nvdash \bot$.

To interpret formulas $\alpha \in \mathcal{L}_{\mathrm{DL}^a}$, Hansson identifies a model[2] with a binary relation R defined over V, so that $\mathsf{R} \subseteq V \times V$. If we think of the members of V as possible worlds, then R may be thought to rank such worlds according to some standard of value, or desirability, or preference, or what have you. Hansson himself takes R to reflect the relation 'is at least as ideal as', [5], p. 143. I will often call such models/relations 'H-models', to contrast them with other models described below.

The strict counterpart, P, of R is specified as usual, For $\varphi, \varphi' \in V$,

- $\varphi \mathsf{P} \varphi'$ iff $\varphi \mathsf{R} \varphi'$ and not-$(\varphi' \mathsf{R} \varphi)$.

Note that P is transitive if R is. In general, however, unless explicitly stated otherwise, we do not assume any particular properties for these relations, except that by its definition P must be asymmetric, hence irreflexive, regardless of the nature of R.

As usual too, for such an R and for $X \subseteq V$, we define the 'maximal' members of X, i.e., maximal with respect to R, or R-maximal, as Hansson did, [5], p. 143:[3]

- $\mathrm{Max}_{\mathsf{R}}(X) = \{\varphi : \varphi \in X$ and there is no $\varphi' \in X$ such that $\varphi' \mathsf{P} \varphi\}$.

The informal idea that B is obligatory under circumstances A just in case B must hold in all the 'best' A-worlds, is now realized in the concept of maximality. Hansson specifies that $O(B/A)$ is true in R just in case all the maximal members of $|A|$ are within $|B|$.

- Rule H $\quad \mathsf{R} \models_{\mathrm{H}} O(B/A)$ iff $\mathrm{Max}_{\mathsf{R}}(|A|) \subseteq |B|$.

From that, the rest of the relation $\models_{\mathrm{H}}$ is as usual:

$$\begin{aligned}
&\mathsf{R} \models_{\mathrm{H}} \neg\alpha \text{ iff } \mathsf{R} \not\models_{\mathrm{H}} \alpha,\\
&\mathsf{R} \models_{\mathrm{H}} \alpha \wedge \beta \text{ iff } \mathsf{R} \models_{\mathrm{H}} \alpha \text{ and } \mathsf{R} \models_{\mathrm{H}} \beta,\\
&\mathsf{R} \models_{\mathrm{H}} \alpha \vee \beta \text{ iff } \mathsf{R} \models_{\mathrm{H}} \alpha \text{ or } \mathsf{R} \models_{\mathrm{H}} \beta,\\
&\mathsf{R} \models_{\mathrm{H}} \alpha \to \beta \text{ iff } \mathsf{R} \not\models_{\mathrm{H}} \alpha \text{ or } \mathsf{R} \models_{\mathrm{H}} \beta.
\end{aligned}$$

By the definition of $P(B/A)$ we also have:

$$\mathsf{R} \models_{\mathrm{H}} P(B/A) \text{ iff } \mathrm{Max}_{\mathsf{R}}(|A|) \cap |B| \neq \varnothing.$$

Let us say, as usual:

- $\alpha \in \mathcal{L}_{\mathrm{DL}^a}$ is H-valid, or simply 'valid', with respect to a class of H-models/relations just in case, for every R in that class, $\mathsf{R} \models_{\mathrm{H}} \alpha$.

- α is an H-consequence of $\Gamma \subseteq \mathcal{L}_{\mathrm{DL}^a}$ with respect to a class of H-models/relations just in case, for every R in that class if, for every $\gamma \in \Gamma$, $\mathsf{R} \models_{\mathrm{H}} \gamma$, then $\mathsf{R} \models_{\mathrm{H}} \alpha$.

We write $\Vdash_{\mathrm{H}} \alpha$ to say that α is H-valid and $\Gamma \Vdash_{\mathrm{H}} \alpha$ to say α is an H-consequence of Γ, both with respect to a class of H-models, given by the context. In a similar vein, let us also say, as usual,

[2] Hansson wrote 'valuation', [5], pp. 142–3, but I will reserve that term for the valuations over $\mathcal{L}_{\mathrm{BL}}$, as above.

[3] In his account of DSDL3, Spohn, [16], p. 239, defines maximality somewhat differently. I discuss that other definition in Section 6.3 below. Until then, we follow Hansson's own specification.

- $\alpha \in \mathcal{L}_{\mathrm{DL}^a}$ is satisfiable in a class of H-models/relations just in case there is an R in that class such that $\mathsf{R} \models_{\mathrm{H}} \alpha$.

- $\Gamma \subseteq \mathcal{L}_{\mathrm{DL}^a}$ is satisfiable in a class of H-models/relations just in case there is an R in that class such that for every $\gamma \in \Gamma$, $\mathsf{R} \models_{\mathrm{H}} \gamma$.

We can now define Hansson's systems, as Hansson did, in terms of classes of such models; cf. [5], p. 144.

- DSDL1 is the set of $\mathcal{L}_{\mathrm{DL}^a}$-formulas, α, that are H-valid with respect to the class of all reflexive relations $\mathsf{R} \subseteq V \times V$;

- DSDL2 is the set of $\mathcal{L}_{\mathrm{DL}^a}$-formulas, α, that are H-valid with respect to the class of all relations $\mathsf{R} \subseteq V \times V$ that are reflexive and that also meet the condition that, for all $A \in \mathcal{L}_{\mathrm{BL}}$, if $|A| \neq \varnothing$, then $\mathrm{Max}_{\mathsf{R}}(|A|) \neq \varnothing$;

- DSDL3 is the set of $\mathcal{L}_{\mathrm{DL}^a}$-formulas, α, that are H-valid with respect to the class of all relations $\mathsf{R} \subseteq V \times V$ that meet the conditions for DSDL2 and for which R is transitive, i.e., for all $\varphi, \varphi', \varphi'' \in V$, if $\varphi \mathsf{R} \varphi'$ and $\varphi' \mathsf{R} \varphi''$ then $\varphi \mathsf{R} \varphi''$, and also total (complete, strongly connected), i.e., for all $\varphi, \varphi' \in V$ either $\varphi \mathsf{R} \varphi'$ or $\varphi' \mathsf{R} \varphi$.

For DSDL1 and DSDL2, reflexivity is nice, but not necessary; the same formulas would be valid with or without this condition; cf. Hansson [5], p. 143. The same is true for DSDL2.5 below. Of course, for DSDL3, totality implies reflexivity, but even that condition could be weakened to weak connectivity, that if $\varphi \neq \varphi'$ then $\varphi \mathsf{R} \varphi'$ or $\varphi' \mathsf{R} \varphi$, without affecting the set of valid formulas, with or without reflexivity. Later, at the end of §5, we will see that transitivity is also an optional condition for relations R for all the logics, except of course DSDL3 where it is explicitly required.

The condition for DSDL2 relations R is a form of the famous, or infamous, Limit Assumption. Here it is a sort of consistency condition, calling for there to be at least one maximal A-world whenever there are in fact A-worlds, i.e., whenever A is consistent. Let us say that R is limited just in case it meets this condition for all $A \in \mathcal{L}_{\mathrm{BL}}$.

- $\mathsf{R} \subseteq V \times V$ is limited iff, for any $A \in \mathcal{L}_{\mathrm{BL}}$, if $|A| \neq \varnothing$, then $\mathrm{Max}_{\mathsf{R}}(|A|) \neq \varnothing$.

Later we will apply that terminology to other relations for much the same condition, and also to models containing such a relation.

While that is a form of the so-called Limit Assumption, it still allows that there could be sequences of A-worlds not capped or limited by any maximal members. Some might increase in rank without end; others might form terminal loops in terms of P so that no member comes out on top. A stronger form of the Limit Assumption would block those possibilities. This is the condition known in the literature on non-monotonic logic as 'stoppering' or 'smoothness' for R.[4] As pertains to the present framework,

- $\mathsf{R} \subseteq V \times V$ is stoppered iff, for every $A \in \mathcal{L}_{\mathrm{BL}}$, if $\varphi \in |A|$, then either $\varphi \in \mathrm{Max}_{\mathsf{R}}(|A|)$ or there is a $\varphi' \in \mathrm{Max}_{\mathsf{R}}(|A|)$ such that $\varphi' \mathsf{P} \varphi$.

[4]See, for example, [7, 8, 9, 14]. In [3] I called this 'being limited' and described the Limit Assumption in terms of it; this should not be confused with the present sense of limitation.

Stoppering implies limitation, but not conversely. If R is total and transitive, however, as for DSDL3, then the two are equivalent; proved in Theorem 3 below.

To capture the class of stoppered relations, we add to Hansson's framework the system I call DSDL2.5 since it is stronger that DSDL2 and weaker than DSDL3; also proved in Theorem 3.

- DSDL2.5 is the set of $\mathcal{L}_{\mathrm{DL}^a}$-formulas, α, that are H-valid with respect to the class of all relations $\mathsf{R} \subseteq V \times V$ that are reflexive and stoppered.

2.3 P-models

Here is another sort of model that is often used, with variations, to interpret formulas of dyadic deontic logic. I call these P-models since, like Hansson's, they are based on preference-like relations. They are more general than his, however.

A P-model is a structure $M = \langle W, \leq, v\rangle$ in which W is a nonempty set of points or so-called possible worlds, and v is a function assigning to each atom $p \in \mathcal{L}_{\mathrm{BL}}$ a set of such points, the worlds where it holds true in M, so that $v(p) \subseteq W$. $\leq$ is a binary relation of relative value, preference, desirability, comparative ideality, or what have you, defined over W such that $\leq \subseteq W \times W$. Except as explicitly stated, we assume no particular properties for $\leq$. The strict counterpart, $<$, of $\leq$ is defined as usual:

- $w < w'$ iff $w \leq w'$ and $w' \not\leq w$.

Formulas $A \in \mathcal{L}_{\mathrm{BL}}$ are interpreted in the usual way, by sets of worlds in which A holds true according to M. Given $M = \langle W, \leq, v\rangle$:

$$
\begin{aligned}
&|p|_M = v(p),\\
&|\neg A|_M = W - |A|_M,\\
&|A \wedge B|_M = |A|_M \cap |B|_M,\\
&|A \vee B|_M = |A|_M \cup |B|_M,\\
&|A \to B|_M = (W - |A|_M) \cup |B|_M.
\end{aligned}
$$

Formulas $O(B/A) \in \mathcal{L}_{\mathrm{DL}^a}$ are interpreted in P-models very much as in Hansson's H-models. With $\mathrm{Max}_{\leq}(|A|_M)$ defined as was $\mathrm{Max}_{\mathsf{R}}(|A|)$:

- Rule P $\quad M \models_{\mathrm{P}} O(B/A)$ iff $\mathrm{Max}_{\leq}(|A|_M) \subseteq |B|_M$.

Extension of $\models_{\mathrm{P}}$ to Boolean constructions, negations, conjunctions, etc., of such deontic formulas, $O(B/A)$, is as always. As with H-models, by the definition of $P(B/A)$, also

$$M \models_{\mathrm{P}} P(B/A) \text{ iff } \mathrm{Max}_{\leq}(|A|_M) \cap |B|_M \neq \varnothing.$$

A formula α is P-valid, or simply 'valid', for a class of P-models, $\Vdash_{\mathrm{P}} \alpha$, just in case for every P-model, M, in that class, $M \models_{\mathrm{P}} \alpha$, and similarly $\Gamma \Vdash_{\mathrm{P}} \alpha$ just in case for every P-model, M, in the class, if $M \models_{\mathrm{P}} \gamma$ for every $\gamma \in \Gamma$, then $M \models_{\mathrm{P}} \alpha$.

A P-model $M = \langle W, \leq, v\rangle$ is limited, or stoppered, just in case its relation $\leq$ is limited, or stoppered, in the sense described above for Hanssonian relations R. For later reference, we note that if W is finite and $\leq$ is transitive, then $\leq$ is necessarily stoppered.

Rule P is clearly much the same as Hansson's Rule H, just as the relation $\leq$ in a P-model corresponds very much to his R. In other respects, however, this framework differs significantly from Hansson's. In P-models, $M = \langle W, \leq, v\rangle$, W can be any sort of nonempty set, finite or infinite, and it can include members that are nothing like the classical valuations that comprise the fields of relations R. Moreover, W with v may have no worlds to correspond to some valuations. In that case, there may be formulas $A \in \mathcal{L}_{\mathrm{BL}}$ that are consistent yet $|A|_M = \varnothing$; in effect, A is consistent but not possible, according to the model. Models where that does not occur will be called 'replete'.

- A P-model $M = \langle W, \leq, v\rangle$ is *replete* for $\mathcal{L}_{\mathrm{BL}}$ just in case, for all $A \in \mathcal{L}_{\mathrm{BL}}$, if $|A| \neq \varnothing$ then $|A|_M \neq \varnothing$.

Furthermore, P-models, $M = \langle W, \leq, v\rangle$, might also have distinct $w, w' \in W$ that agree on all formulas $A \in \mathcal{L}_{\mathrm{BL}}$, and so are indistinguishable duplicates of each other, as far as M is concerned. Yet, though indistinguishable, w and w' remain distinct and may stand in different positions vis-à-vis $\leq$. Let us say, for a given $M = \langle W, \leq, v\rangle$,

- $w, w' \in W$ are duplicates of each other in M just in case, $w \neq w'$ and, for all $A \in \mathcal{L}_{\mathrm{BL}}$, $w \in |A|_M$ iff $w' \in |A|_M$.

P-models with duplicates are 'redundant'; those without are 'irredundant'.

- $M = \langle W, \leq, v\rangle$ is *redundant* just in case there are $w, w' \in W$ that are duplicates of each other in M.

- $M = \langle W, \leq, v\rangle$ is *irredundant* just in case it is not redundant.

By extension, Hanssonian models/relations R over V may also be said to be irredundant since necessarily V contains no duplicates.

3 Axiomatics

The preceding section presented Hansson's semantically defined DSDL logics, as well as DSDL2.5. Here we introduce the axiomatic systems, DDL-a, DDL-b, DDL-c and DDL-d, that will be proved to be equivalent to those semantical systems. All contain the classical propositional calculus over $\mathcal{L}_{\mathrm{DL}^a}$, including closure under its rules; they then add all instances of the following schemas:

- For DDL-a:

  (LLE) $O(C/A) \leftrightarrow O(C/B)$ when $\vdash A \leftrightarrow B$,
  (RW) $O(B/A) \rightarrow O(C/A)$ when $\vdash B \rightarrow C$,
  (Reflex) $O(A/A)$,
  (AND) $(O(B/A) \wedge O(C/A)) \rightarrow O(B \wedge C/A)$,
  (OR) $(O(C/A) \wedge O(C/B)) \rightarrow O(C/A \vee B)$.

- DDL-b = DDL-a + all instances of:

  (RP) $P(\top/A)$ when A is BL-consistent.

- DDL-c = DDL-b + all instances of Cautious Monotony:

  (CautMono) $(O(B/A) \wedge O(C/A)) \rightarrow O(C/A \wedge B)$.

- DDL-d = DDL-b + all instances of Rational Monotony:

  (RatMono) $(P(B/A) \wedge O(C/A)) \rightarrow O(C/A \wedge B)$.

We note that (CautMono) is derivable in DDL-d, and thus DDL-d is an extension of DDL-c; see the proof of Theorem 3 below.

The usual definitions of derivation, derivability, theorem, etc. apply to each of these. In the postulates (LLE) and (RW) I write $\vdash$ to indicate derivability within the classical propositional calculus for $\mathcal{L}_{\mathrm{BL}}$. I will also write $\vdash$ to indicate derivability within the DDL systems. No confusion should result from this ambiguity, given the divide between $\mathcal{L}_{\mathrm{BL}}$ and $\mathcal{L}_{\mathrm{DL}^a}$, and the notation we use for the formulas of each. For each axiomatic system, **L**, α is consistent for **L**, or simply consistent when **L** is given in context, when, as usual, $\nvdash \neg\alpha$ in **L**, and similarly, $\Gamma \subseteq \mathcal{L}_{\mathrm{DL}^a}$ is consistent for **L** when $\Gamma \nvdash \beta \wedge \neg\beta$ in **L**, for any $\beta \in \mathcal{L}_{\mathrm{DL}^a}$.

In the following sections I will demonstrate that

- DDL-a = DSDL1,
- DDL-b = DSDL2,
- DDL-c = DSDL2.5,
- DDL-d = DSDL3.

Half of that is easy; it amounts to the soundness of the axiomatic systems.

Theorem 1 *(i) DDL-a is sound with respect to the class of all reflexive H-models* R. *(ii) DDL-b is sound with respect to the class of all H-models* R *that are reflexive and limited. (iii) DDL-c is sound with respect to the class of all H-models* R *that are reflexive and stoppered. (iv) DDL-d is sound with respect to the class of all H-models* R *that are reflexive, transitive and total and also limited. I.e., if* $\vdash \alpha$ *in one of these systems, then* $\Vdash_{\mathrm{H}} \alpha$ *with respect to the appropriate class of H-models.*

Proof. Proved in the usual way by showing that all axioms are valid for the respective classes of models and the rules of the logics preserve validity. These are easy enough to leave to the reader, though for illustration we present the validity of (CautMono) and (RatMono) with respect to DSDL2.5 and DSDL3 models.

For (CautMono), suppose R is stoppered, and that $\mathsf{R} \models_{\mathrm{H}} O(B/A)$ and $\mathsf{R} \models_{\mathrm{H}} O(C/A)$, so that $\mathrm{Max}_{\mathsf{R}}(|A|) \subseteq |B|$ and $\mathrm{Max}_{\mathsf{R}}(|A|) \subseteq |C|$. Suppose $\varphi \in \mathrm{Max}_{\mathsf{R}}(|A \wedge B|)$. Since $\varphi \in |A|$ and R is stoppered, $\varphi \in \mathrm{Max}_{\mathsf{R}}(|A|)$ or there is a $\varphi' \in \mathrm{Max}_{\mathsf{R}}(|A|)$ such that $\varphi' \mathsf{P} \varphi$. The second is not possible, for if it were then $\varphi' \in |B|$, whence $\varphi' \in |A \wedge B|$, in which case $\varphi \notin \mathrm{Max}_{\mathsf{R}}(|A \wedge B|)$, a contradiction. Therefore, $\varphi \in \mathrm{Max}_{\mathsf{R}}(|A|)$ and so $\varphi \in |C|$. That suffices for $\mathrm{Max}_{\mathsf{R}}(|A \wedge B|) \subseteq |C|$, and so for $\mathsf{R} \models_{\mathrm{H}} O(C/A \wedge B)$.

For (RatMono), suppose R is transitive, total and limited, and that $\mathsf{R} \models_{\mathrm{H}} P(B/A)$ and $\mathsf{R} \models_{\mathrm{H}} O(C/A)$, so that there is some $\varphi \in \mathrm{Max}_{\mathsf{R}}(|A|)$ such that $\varphi \in |B|$, and also $\mathrm{Max}_{\mathsf{R}}(|A|) \subseteq |C|$. Consider any $\varphi' \in \mathrm{Max}_{\mathsf{R}}(|A \wedge B|)$; we show $\varphi' \in \mathrm{Max}_{\mathsf{R}}(|A|)$. Since

$\varphi' \in |A|$, suppose, for *reductio*, there is some $\varphi'' \in |A|$ such that $\varphi''\mathsf{P}\varphi'$. By totality, either $\varphi\mathsf{R}\varphi''$ or both $\varphi''\mathsf{R}\varphi$ and not-($\varphi\mathsf{R}\varphi''$), i.e., $\varphi''\mathsf{P}\varphi$. The second case is not possible, however, since then $\varphi \notin \mathrm{Max}_{\mathsf{R}}(|A|)$, a contradiction. Hence, $\varphi\mathsf{R}\varphi''$. By transitivity, $\varphi\mathsf{P}\varphi'$. In that case, since $\varphi \in |A \wedge B|$, $\varphi' \notin \mathrm{Max}_{\mathsf{R}}(|A \wedge B|)$, another contradiction. Therefore, there is no such φ'' and so $\varphi' \in \mathrm{Max}_{\mathsf{R}}(|A|)$. Then $\varphi' \in |C|$. That suffices for $\mathrm{Max}_{\mathsf{R}}(|A \wedge B|) \subseteq |C|$, and so for $\mathsf{R} \models_{\overline{\mathrm{H}}} O(C/A \wedge B)$. ■

Corollary 2 *DDL-a ⊆ DSDL1; DDL-b ⊆ DSDL2; DDL-c ⊆ DSDL2.5; DDL-d ⊆ DSDL3.*

Theorem 1 is of fundamental importance, of course. It is also useful to demonstrate that the systems are indeed separate, and each is a proper extension of its alphabetical or numerical predecessors.

Theorem 3 *(i) DDL-a ⊂ DDL-b ⊂ DDL-c ⊂ DDL-d. (ii) DSDL1 ⊂ DSDL2 ⊂ DSDL2.5 ⊂ DSDL3.*

Proof. That DDL-a ⊆ DDL-b, etc. is obvious, except, perhaps, for DDL-c ⊆ DDL-d and DSDL2.5 ⊆ DSDL3. For the former, we now show that (CautMono) is derivable in DDL-d. Suppose $O(B/A)$ and $O(C/A)$. Either $P(B/A)$ or $\neg P(B/A)$. If the first, then $O(C/A \wedge B)$ by (RatMono). If the second, then $O(\neg B/A)$ by definition; hence, $O(B \wedge \neg B/A)$ by (AND). So $O(\perp/A)$. In that case, by (RP), A must be inconsistent, and $\vdash A \leftrightarrow \perp$, whence $\vdash A \leftrightarrow (A \wedge B)$, from which $O(C/A \wedge B)$ follows by (LLE).

For DSDL2.5 ⊆ DSDL3, it suffices that if R is limited, transitive, and total, then R is stoppered. To see that, suppose $\varphi \in |A|$. Since R is limited, there is a $\varphi' \in \mathrm{Max}_{\mathsf{R}}(|A|)$. Since R is total, either $\varphi\mathsf{R}\varphi'$ or $\varphi'\mathsf{P}\varphi$. In the first case, $\varphi \in \mathrm{Max}_{\mathsf{R}}(|A|)$, for if not, then there is a $\varphi'' \in |A|$ such that $\varphi''\mathsf{P}\varphi$, in which case $\varphi''\mathsf{P}\varphi'$, since R is transitive, and then $\varphi' \notin \mathrm{Max}_{\mathsf{R}}(|A|)$, a contradiction. Thus, in this case, $\varphi \in \mathrm{Max}_{\mathsf{R}}(|A|)$, which suffices for stoppering. In the second case, since already $\varphi' \in \mathrm{Max}_{\mathsf{R}}(|A|)$, there is a $\varphi' \in \mathrm{Max}_{\mathsf{R}}(|A|)$ such that $\varphi'\mathsf{P}\varphi$, which also suffices for stoppering.

To show the listed containments to be proper, we now give (a) an instance of (RP) not valid for DSDL1, (b) an instance of (CautMono) not valid for DSDL2 and (c) an instance of (RatMono) not valid for DSDL2.5. For all of these we distinguish four valuations, which might be thought of as determining the first four rows of a conventional truth-table for $p, q, r \in \mathcal{L}_{\mathrm{BL}}$. Let $\mathbb{S}$ be the set of all atoms of $\mathcal{L}_{\mathrm{BL}}$ other than p, q, r. Let $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ be those members of V such that:

$$\begin{array}{lll}
\varphi_1(p) = 1 & \varphi_1(q) = 1 & \varphi_1(r) = 1 \\
\varphi_2(p) = 1 & \varphi_2(q) = 1 & \varphi_2(r) = 0 \\
\varphi_3(p) = 1 & \varphi_3(q) = 0 & \varphi_3(r) = 1 \\
\varphi_4(p) = 1 & \varphi_4(q) = 0 & \varphi_4(r) = 0
\end{array}$$
$$\varphi_1(s) = \varphi_2(s) = \varphi_3(s) = \varphi_4(s) = 1, \text{ for all other atoms } s \in \mathbb{S}.$$

Let Y be the set of all valuations $\varphi \in V$ other than $\varphi_1, \varphi_2, \varphi_3, \varphi_4$.

(a) Consider this instance of (RP): $P(\top/p)$, since atom p is consistent. Let R_1 be exactly such that for all $\varphi \in V$, $\varphi\mathsf{R}_1\varphi$, and also $\varphi_1\mathsf{R}_1\varphi_2$, $\varphi_2\mathsf{R}_1\varphi_3$, $\varphi_3\mathsf{R}_1\varphi_4$, $\varphi_4\mathsf{R}_1\varphi_1$, and $\varphi_4\mathsf{R}_1\varphi$, for all other $\varphi \in Y$. R_1 is obviously reflexive. Also, the four selected valuations form a loop by P_1 and φ_4 ranks higher than all other valuations. Because of that and the loop, $\mathrm{Max}_{\mathsf{R}_1}(|p|) = \varnothing$. Hence, $\mathsf{R}_1 \not\models_{\mathrm{H}} P(\top/p)$. Therefore, $P(\top/p) \notin$ DSDL1 and $P(\top/p) \notin$ DDL-a.

(b) Consider this instance of (CautMono): $(O(q/p) \wedge O(r/p)) \rightarrow O(r/p \wedge q)$. Let R_2 be exactly such that for all $\varphi \in V$, $\varphi\mathsf{R}_2\varphi$, and, for all $\varphi \in Y$, $\varphi_1\mathsf{R}_2\varphi$, while $\varphi_2\mathsf{R}_2\varphi_4$, $\varphi_4\mathsf{R}_2\varphi_3$ and $\varphi_3\mathsf{R}_2\varphi_2$. Thus, $\varphi_2, \varphi_3, \varphi_4$ form a loop by P_2 while φ_1 stands alone, though ranked higher than all $\varphi \in Y$. By inspection, $\mathrm{Max}_{\mathsf{R}_2}(|p|) = \{\varphi_1\}$. Since $\varphi_1 \in |q|$ and $\varphi_1 \in |r|$, $\mathrm{Max}_{\mathsf{R}_2}(|p|) \subseteq |q|$ and $\mathrm{Max}_{\mathsf{R}_2}(|p|) \subseteq |r|$. Hence, $\mathsf{R}_2 \models_{\mathrm{H}} O(q/p)$ and $\mathsf{R}_2 \models_{\mathrm{H}} O(r/p)$. On the other hand, $\mathrm{Max}_{\mathsf{R}_2}(|p \wedge q|) = \{\varphi_1, \varphi_2\}$, and since $\varphi_2 \notin |r|$, $\mathrm{Max}_{\mathsf{R}_2}(|p \wedge q|) \nsubseteq |r|$, so that $\mathsf{R}_2 \not\models_{\mathrm{H}} O(r/p \wedge q)$. It remains to show that R_2 is limited. Consider any $A \in \mathcal{L}_{\mathrm{BL}}$ such that $|A| \neq \varnothing$. If $\varphi_1 \in |A|$, then $\varphi_1 \in \mathrm{Max}_{\mathsf{R}_2}(|A|)$, so that $\mathrm{Max}_{\mathsf{R}_2}(|A|) \neq \varnothing$. Suppose then that $\varphi_1 \notin |A|$. It follows that, for all $\varphi \in Y$, if $\varphi \in |A|$ then $\varphi \in \mathrm{Max}_{\mathsf{R}_2}(|A|)$, and so $\mathrm{Max}_{\mathsf{R}_2}(|A|) \neq \varnothing$. Suppose $\varphi_2 \in |A|$. Let $\mathbb{S}(A)$ be the set of all atoms of $\mathbb{S}$ that are in A. Consider φ^* such that $\varphi^*(p) = \varphi^*(q) = 1$ and $\varphi^*(r) = 0$ and $\varphi^*(s) = 1$ for all $s \in \mathbb{S}(A)$ and $\varphi^*(t) = 0$ for all other atoms $t \in \mathbb{S} - \mathbb{S}(A)$. There must be one such $\varphi^* \in V$. For it, φ^* agrees with φ_2 on all atoms that are constituents of A. Hence, $\varphi^* \in |A|$. But $\varphi^* \neq \varphi_2$, nor φ_3, nor φ_4, nor φ_1. Hence, $\varphi^* \in Y$, and so $\varphi^* \in \mathrm{Max}_{\mathsf{R}_2}(|A|)$ and $\mathrm{Max}_{\mathsf{R}_2}(|A|) \neq \varnothing$. In case $\varphi_3 \in |A|$ or $\varphi_4 \in |A|$, argue similarly that $\mathrm{Max}_{\mathsf{R}_2}(|A|) \neq \varnothing$. In case none of φ_1–φ_4 are in $|A|$, then there must be some $\varphi \in Y$ such that $\varphi \in |A|$, in which case again $\mathrm{Max}_{\mathsf{R}_2}(|A|) \neq \varnothing$. Hence, R_2 is limited. Thus, $(O(q/p) \wedge O(r/p)) \rightarrow O(r/p \wedge q) \notin$ DSDL2, and so $(O(q/p) \wedge O(r/p)) \rightarrow O(r/p \wedge q) \notin$ DDL-b.

(c) Consider this instance of (RatMono): $(P(q/p) \wedge O(r/p)) \rightarrow O(r/p \wedge q)$. Let R_3 be exactly such that for all $\varphi \in V$, $\varphi\mathsf{R}_3\varphi$; also, for all $\varphi \in Y$, $\varphi_1\mathsf{R}_3\varphi$; and $\varphi_1\mathsf{R}_3\varphi_4$ and $\varphi_3\mathsf{R}_3\varphi_2$. Thus, $\mathrm{Max}_{\mathsf{R}_3}(|p|) = \{\varphi_1, \varphi_3\}$. Since both are in $|r|$, $\mathrm{Max}_{\mathsf{R}_3}(|p|) \subseteq |r|$, so that $\mathsf{R}_3 \models_{\mathrm{H}} O(r/p)$. Also $\mathrm{Max}_{\mathsf{R}_3}(|p|) \cap |q| \neq \varnothing$, by virtue of φ_1. Hence, $\mathsf{R}_3 \models_{\mathrm{H}} P(q/p)$. On the other hand, $\mathrm{Max}_{\mathsf{R}_3}(|p \wedge q|) = \{\varphi_1, \varphi_2\}$, and so $\mathrm{Max}_{\mathsf{R}_3}(|p \wedge q|) \nsubseteq |r|$, by virtue of φ_2. Hence, $\mathsf{R}_3 \not\models_{\mathrm{H}} O(r/p \wedge q)$. We show that R_3 is stoppered. Suppose some $A \in \mathcal{L}_{\mathrm{BL}}$ and some φ such that $\varphi \in |A|$. If $\varphi = \varphi_1$ then $\varphi \in \mathrm{Max}_{\mathsf{R}_3}(|A|)$. Similarly if $\varphi = \varphi_3$. If $\varphi = \varphi_2$, then if $\varphi_3 \in |A|$ then $\varphi_3 \in \mathrm{Max}_{\mathsf{R}_3}(|A|)$ and $\varphi_3\mathsf{P}_3\varphi$. If $\varphi_3 \notin |A|$, then $\varphi \in \mathrm{Max}_{\mathsf{R}_3}(|A|)$. Similarly if $\varphi = \varphi_4$, with φ_1 in place of φ_3. Likewise, if $\varphi \in Y$. Hence, for any A and φ, if $\varphi \in |A|$, then either $\varphi \in \mathrm{Max}_{\mathsf{R}_3}(|A|)$ or there is a $\varphi' \in \mathrm{Max}_{\mathsf{R}_3}(|A|)$ such that $\varphi'\mathsf{P}_3\varphi$, which is to say, R_3 is stoppered. Consequently, $(P(q/p) \wedge O(r/p)) \rightarrow O(r/p \wedge q) \notin$ DSDL2.5, and $(P(q/p) \wedge O(r/p)) \rightarrow O(r/p \wedge q) \notin$ DDL-c. ■

4 DDL-d = DSDL3

As we establish the equivalences of the axiomatic systems with Hansson's semantically defined dyadic deontic logics, we begin with DDL-d and DSDL3 because that is the most straight-forward. The demonstration will follow familiar paths, and thus perhaps be a comfortable exercise to limber up for more arduous hikes to come.

Since we already have the soundness of DDL-d in Theorem 1, let us turn straight away to completeness. Let w be any maximal DDL-d consistent set of formulas of $\mathcal{L}_{\mathrm{DL}^a}$. Given w, for any $A \in \mathcal{L}_{\mathrm{BL}}$, let

- $\Delta_A = \{B : O(B/A) \in \mathsf{w}\}$.

Lemma 4 *(i) If* $P(\top/A) \in \mathsf{w}$ *then* Δ_A *is BL-consistent. (ii) If* $O(B/A) \notin \mathsf{w}$*, then* $\Delta_A \cup \{\neg B\}$ *is BL-consistent.*

Proof. For (i), suppose $P(\top/A) \in \mathsf{w}$, but that Δ_A is not BL-consistent. Then there are $D_1, \ldots, D_n \in \Delta_A$ such that $\vdash (D_1 \wedge \cdots \wedge D_n) \to \bot$, where for each D_i, $O(D_i/A) \in \mathsf{w}$. By application of (AND) it follows that $O(D_1 \wedge \cdots \wedge D_n/A) \in \mathsf{w}$; it also follows by (RW) that $\vdash O(D_1 \wedge \cdots \wedge D_n/A) \to O(\bot/A)$. Hence, $O(\bot/A) \in \mathsf{w}$. Since $P(\top/A) \in \mathsf{w}$, $\neg O(\neg\top/A) \in \mathsf{w}$, i.e., $\neg O(\bot/A) \in \mathsf{w}$, contrary to the consistency of w. Hence, Δ_A must be consistent. The argument for (ii) is similar. ■

This lemma will apply as well in later arguments for the other systems.

As earlier, V is the set of classical valuations for $\mathcal{L}_{\mathrm{BL}}$. Given w, let us say, for $\varphi \in V$ and $A \in \mathcal{L}_{\mathrm{BL}}$:

- φ is w-normal for A iff for all $B \in \Delta_A$, $\varphi \in |B|$.

Henceforth, we drop the prefix 'w' on 'w-normal', and just say 'normal'.

We now define an H-model, i.e., a binary relation $\mathsf{R} \subseteq V \times V$, that will be shown to verify all and only sentences in w, Lemma 8 below.

- For $\varphi, \varphi' \in V$, $\varphi \mathsf{R} \varphi'$ iff for all B such that φ' is normal for B, there is an A such that φ is normal for A and $P(A/A \vee B) \in \mathsf{w}$.

To demonstrate that R has the requisite properties for a DSDL3 relation and that it does the work it is supposed to do, it is helpful to know the following principles are derivable in DDL-d. The first, often called (S), is derivable even in DDL-a, and so in all the DDL systems; it will figure frequently in the proofs for those other systems. (S) is, in fact, interderivable with (OR), given the other postulates of DDL-a. Here we derive it using (OR), and leave it as an exercise to the reader to derive (OR) from it. It is not difficult. The second principle applies to showing that R is transitive; it is interderivable with (RatMono). We derive (transit) here, and leave the derivation of (RatMono) from it as another exercise. It is harder. The third principle is less interesting; it applies to showing that R is total.

Lemma 5 *These are derivable in DDL-d:*

(1)	*(S)*	$O(C/A \wedge B) \to O(B \to C/A)$,
(2)	*(transit)*	$(P(A/A \vee B) \wedge P(B/B \vee C)) \to P(A/A \vee C)$,
(3)	*(total)*	$P(\top/A \vee B) \to (P(A/A \vee B) \vee P(B/A \vee B))$.

Proof. (1) For (S), suppose $O(C/A \wedge B)$. Then $O(B \to C/A \wedge B)$, by (RW). Also, $O(A \wedge \neg B/A \wedge \neg B)$, by (Reflex); hence, $O(B \to C/A \wedge \neg B)$, also by (RW). From those, $O(B \to C/(A \wedge B) \vee (A \wedge \neg B))$, by (OR), and then $O(B \to C/A)$ follows by (LLE).

(2) For (transit), suppose (a) $P(A/A \vee B)$ and (b) $P(B/B \vee C)$, i.e., $\neg O(\neg A/A \vee B)$ and $\neg O(\neg B/B \vee C)$. Suppose, for *reductio*, (c) $\neg P(A/A \vee C)$, i.e., $O(\neg A/A \vee C)$. Then $O(\neg A/(A \vee C) \wedge (A \vee B \vee C))$ by (LLE), so $O((A \vee C) \to \neg A/A \vee B \vee C)$ by (S), i.e., (1) above. From that, (i) $O(\neg A/A \vee B \vee C)$ follows by (RW). We now derive (ii) $P(A \vee B/A \vee B \vee C)$, i.e., $\neg O(\neg(A \vee B)/A \vee B \vee C)$, by another *reductio*. For that, suppose

(d) $O(\neg(A \vee B)/A \vee B \vee C)$. Under that supposition, we want (iii) $P(B \vee C/A \vee B \vee C)$, and so suppose, for a third *reductio*, (e) $O(\neg(B \vee C)/A \vee B \vee C)$. Since $O(A \vee B \vee C/A \vee B \vee C)$, (Reflex), $O((A \vee B \vee C) \wedge \neg(B \vee C))/A \vee B \vee C)$ by (AND), whence $O(A/A \vee B \vee C)$ by (RW). With (i) that yields $O(A \wedge \neg A/A \vee B \vee C)$ by (AND), and so $O(\bot/A \vee B \vee C)$. Given (RP), it follows that $A \vee B \vee C$ is BL-inconsistent, i.e., $\vdash (A \vee B \vee C) \leftrightarrow \bot$. From that, $\vdash (A \vee B \vee C) \leftrightarrow (A \vee B)$. Then, from (i) $O(\neg A/A \vee B)$ by (LLE), contrary to (a) above. Therefore, (e) is false; so (iii) $P(B \vee C/A \vee B \vee C)$. From (d), it follows that $O(\neg B/A \vee B \vee C)$ by (RW). With (iii), $O(\neg B/(B \vee C) \wedge (A \vee B \vee C))$ by (RatMono). Thence $O(\neg B/B \vee C)$ by (LLE). That contradicts (b) above. Hence, (d) is false, and so (ii) $P(A \vee B/A \vee B \vee C)$. (i) and (ii) yield $O(\neg A/(A \vee B) \wedge (A \vee B \vee C))$ by (RatMono). Then $O(\neg A/A \vee B)$ by (LLE). That contradicts (a). Hence, (c) too is false, and therefore, $P(A/A \vee C)$, as required.

(3) For (total), suppose $P(\top/A \vee B)$, i.e., $\neg O(\bot/A \vee B)$, and suppose $\neg P(A/A \vee B)$ and $\neg P(B/A \vee B)$, i.e., $O(\neg A/A \vee B)$ and $O(\neg B/A \vee B)$. Then $O(\neg A \wedge \neg B/A \vee B)$ by (AND). Since $O(A \vee B/A \vee B)$, (Reflex), $O((A \vee B) \wedge \neg A \wedge \neg B/A \vee B)$ by (AND) again, and thus $O(\bot/A \vee B)$ by (RW), which contradicts the first supposition. Hence, either $P(A/A \vee B)$ or $P(B/A \vee B)$; so, if $P(\top/A \vee B)$, then $P(A/A \vee B) \vee P(B/A \vee B)$. ■

Lemma 6 R *is reflexive, transitive, and total.*

Proof. Reflexivity follows from totality below. For transitivity, suppose $\varphi \mathsf{R} \varphi'$ and $\varphi' \mathsf{R} \varphi''$. We show $\varphi \mathsf{R} \varphi''$. For that, consider any C such that φ'' is normal for C. Since $\varphi' \mathsf{R} \varphi''$, there is a B such that φ' is normal for B and $P(B/B \vee C) \in \mathsf{w}$. Since $\varphi \mathsf{R} \varphi'$, there is an A such that φ is normal for A and $P(A/A \vee B) \in \mathsf{w}$. By (transit), Lemma 5(2), $P(A/A \vee C) \in \mathsf{w}$. That suffices for $\varphi \mathsf{R} \varphi''$, as required.

For totality, consider any $\varphi, \varphi' \in V$. Suppose it is not the case that $\varphi \mathsf{R} \varphi'$. Then there must be some B such that φ' is normal for B and for all A such that φ is normal for A, $P(A/A \vee B) \notin \mathsf{w}$. To see that $\varphi' \mathsf{R} \varphi$, consider any C such that φ is normal for C. Then $P(C/C \vee B) \notin \mathsf{w}$. Since φ' is normal for B, $\varphi' \in |B|$, and so $\varphi' \in |B \vee C|$. Hence, $B \vee C$ is consistent, and so, by (RP), $\vdash P(\top/B \vee C)$. Thus $P(\top/B \vee C) \in \mathsf{w}$. Given that, and $P(C/B \vee C) \notin \mathsf{w}$, it follows that $P(B/B \vee C) \in \mathsf{w}$ by (total), Lemma 5(3). Hence, for any C such that φ is normal for C there is a B such that φ' is normal for B and $P(B/B \vee C) \in \mathsf{w}$. That suffices for $\varphi' \mathsf{R} \varphi$, as required. ■

Lemma 7 *For all $\varphi \in V$, and all $A \in \mathcal{L}_{\mathrm{BL}}$, φ is normal for A iff $\varphi \in \mathrm{Max}_{\mathsf{R}}(|A|)$.*

Proof. L $\rightarrow$ R: Suppose φ is normal for A. By (Reflex) $O(A/A) \in \mathsf{w}$, so $A \in \Delta_A$. Hence, $\varphi \in |A|$. Suppose then, for *reductio*, $\varphi \notin \mathrm{Max}_{\mathsf{R}}(|A|)$. Then there is some $\varphi' \in |A|$ such that $\varphi' \mathsf{P} \varphi$. Since it is not the case that $\varphi \mathsf{R} \varphi'$, there must be some B such that φ' is normal for B and for all C if φ is normal for C then $P(C/C \vee B) \notin \mathsf{w}$. Hence, $P(A/A \vee B) \notin \mathsf{w}$, i.e., $O(\neg A/A \vee B) \in \mathsf{w}$. Since $\varphi \in |A|$, $\varphi \in |A \vee B|$; hence, $A \vee B$ is consistent. By (RP), $\vdash P(\top/A \vee B)$, so that $P(\top/A \vee B) \in \mathsf{w}$. Since $P(A/A \vee B) \notin \mathsf{w}$, then, by (total), Lemma 5(3), $P(B/A \vee B) \in \mathsf{w}$. Since $O(\neg A/A \vee B) \in \mathsf{w}$, $O(\neg A/(A \vee B) \wedge B) \in \mathsf{w}$ by (RatMono), and then $O(\neg A/B) \in \mathsf{w}$, by (LLE). Since φ' is normal for B, $\varphi' \in |\neg A|$, hence, $\varphi' \notin |A|$. But it is given that $\varphi' \in |A|$; thus a contradiction. Consequently, it must be that $\varphi \in \mathrm{Max}_{\mathsf{R}}(|A|)$.

R $\to$ L: Suppose $\varphi \in \mathrm{Max}_{\mathsf{R}}(|A|)$. Since obviously $\varphi \in |A|$, A is BL-consistent. That being so, $\vdash P(\top/A)$, by (RP), so that $P(\top/A) \in \mathsf{w}$. By Lemma 4(i), Δ_A is BL-consistent. Hence there is a valuation, $\varphi' \in V$ such that for all $C \in \Delta_A$, $\varphi' \in |C|$. φ' is thus normal for A; so also $\varphi' \in |A|$. Since $\varphi \in \mathrm{Max}_{\mathsf{R}}(|A|)$ and R is total, Lemma 6, $\varphi \mathsf{R} \varphi'$. Hence, for all C such that φ' is normal for C, there is some B such that φ is normal for B and $P(B/B \vee C) \in \mathsf{w}$. Hence, there is some such B such that $P(B/A \vee B) \in \mathsf{w}$. Now consider any $D \in \Delta_A$, so that $O(D/A) \in \mathsf{w}$. Then $O(D/A \wedge (A \vee B)) \in \mathsf{w}$, by (LLE); hence, $O(A \to D/A \vee B) \in \mathsf{w}$, by principle (S), Lemma 5(1). It follows that $O(A \to D/(A \vee B) \wedge B) \in \mathsf{w}$, by (RatMono). Hence, $O(A \to D/B) \in \mathsf{w}$, by (LLE). Since φ is normal for B, $\varphi \in |A \to D|$. Since also $\varphi \in |A|$, $\varphi \in |D|$. That suffices for φ to be normal for A, as required. ■

We will see that sort of equivalence between normality and maximality in the proofs to come for the other systems as well.

Lemma 8 *For all $\alpha \in \mathcal{L}_{\mathrm{DL}^a}$, $\alpha \in \mathsf{w}$ iff $\mathsf{R} \models_{\mathrm{H}} \alpha$.*

Proof. By induction on α. I will show only the basis case where $\alpha = O(B/A)$ for some $A, B \in \mathcal{L}_{\mathrm{BL}}$. The induction to Boolean combinations of such formulas is very easy, and left to the reader.

L $\to$ R: Suppose $O(B/A) \in \mathsf{w}$. To show $\mathsf{R} \models_{\mathrm{H}} O(B/A)$, i.e, $\mathrm{Max}_{\mathsf{R}}(|A|) \subseteq |B|$, consider any $\varphi \in \mathrm{Max}_{\mathsf{R}}(|A|)$. By Lemma 7, φ is normal for A. Since $B \in \Delta_A$, $\varphi \in |B|$. That suffices for $\mathrm{Max}_{\mathsf{R}}(|A|) \subseteq |B|$, and so $\mathsf{R} \models_{\mathrm{H}} O(B/A)$.

R $\to$ L: Suppose $\mathsf{R} \models_{\mathrm{H}} O(B/A)$, i.e, $\mathrm{Max}_{\mathsf{R}}(|A|) \subseteq |B|$. Suppose $O(B/A) \notin \mathsf{w}$. By Lemma 4(ii), $\Delta_A \cup \{\neg B\}$ is BL-consistent, and so there is a φ such that for all $C \in \Delta_A \cup \{\neg B\}$, $\varphi \in |C|$. φ is thus normal for A. Hence, $\varphi \in \mathrm{Max}_{\mathsf{R}}(|A|)$, by Lemma 7. So $\varphi \in |B|$. But also $\varphi \in |\neg B|$, or $\varphi \notin |B|$, a contradiction. Hence, $O(B/A) \in \mathsf{w}$. ■

Lemma 9 *R is limited; i.e., for all $A \in \mathcal{L}_{\mathrm{BL}}$, if $|A| \neq \varnothing$, then $\mathrm{Max}_{\mathsf{R}}(|A|) \neq \varnothing$.*

Proof. Suppose $|A| \neq \varnothing$. Then A is BL-consistent. By (RP), $\vdash P(\top/A)$, so $P(\top/A) \in \mathsf{w}$. By Lemma 4(i), Δ_A is BL-consistent. Hence there is a $\varphi \in V$ such that for all $B \in \Delta_A$, $\varphi \in |B|$. φ is thus normal for A. Hence, by Lemma 7, $\varphi \in \mathrm{Max}_{\mathsf{R}}(|A|)$, and so $\mathrm{Max}_{\mathsf{R}}(|A|) \neq \varnothing$. ■

From these lemmas, the strong completeness of DDL-d easily follows.

Theorem 10 *DDL-d is strongly complete with respect to the class of DSDL3 models/relations R; i.e., if $\Gamma \Vdash_{\mathrm{H}} \alpha$ with respect to that class, then $\Gamma \vdash \alpha$ in DDL-d.*

Proof. Suppose $\Gamma \Vdash_{\mathrm{H}} \alpha$ with respect to the class of DSDL3 models/relations R, and suppose, for *reductio*, $\Gamma \nvdash \alpha$. Then $\Gamma \cup \{\neg\alpha\}$ is consistent for DDL-d, and so it has a maximal consistent extension w, by the usual arguments. Given one such w, define the relation R as described above. By Lemmas 6 and 9, R is reflexive, transitive and total, and limited. Hence, R is a DSDL3 model/relation. Since w is an extension of $\Gamma \cup \{\neg\alpha\}$, for every $\gamma \in \Gamma$, $\gamma \in \mathsf{w}$. Hence by Lemma 8, for every $\gamma \in \Gamma$, $\mathsf{R} \models_{\mathrm{H}} \gamma$. Also, since $\neg\alpha \in \mathsf{w}$, $\mathsf{R} \models_{\mathrm{H}} \neg\alpha$. Consequently, $\mathsf{R} \not\models_{\mathrm{H}} \alpha$, and so $\Gamma \nVdash_{\mathrm{H}} \alpha$ with respect to this class of models/relations, a contradiction. Therefore, $\Gamma \vdash \alpha$ in DDL-d. ■

Since strong completeness implies weak completeness, i.e., that all valid formulas are derivable, or, for all $\alpha \in \mathcal{L}_{\mathrm{DL}^a}$, if $\Vdash_{\overline{\mathrm{H}}} \alpha$ then $\vdash \alpha$, then

Corollary 11 *DSDL3* $\subseteq$ *DDL-d.*

Hence, with Theorem 1 and its corollary,

Corollary 12 *DDL-d* = *DSDL3.*

Moreover, since, as is well-known, strong completeness with soundness implies compactness for a class of models, i.e., that, for any Γ, if every finite subset of Γ is satisfiable in the class then Γ is satisfiable in that class, or equivalently, if $\Gamma \Vdash_{\overline{\mathrm{H}}} \alpha$ then there is a finite set $\Gamma_f \subseteq \Gamma$ such that $\Gamma_f \Vdash_{\overline{\mathrm{H}}} \alpha$,

Corollary 13 *DSDL3 is compact.*

Proof. Suppose every finite subset of Γ is satisfiable in DSDL3 models, but that Γ is not so satisfiable. Then, vacuously, $\Gamma \Vdash_{\overline{\mathrm{H}}} \beta \wedge \neg\beta$. By strong completeness, $\Gamma \vdash \beta \wedge \neg\beta$ in DDL-d. By definition of derivability, there is a finite $\Gamma_f \subseteq \Gamma$ such that $\Gamma_f \vdash \beta \wedge \neg\beta$. By soundness, $\Gamma_f \Vdash_{\overline{\mathrm{H}}} \beta \wedge \neg\beta$. Since, by assumption, Γ_f is satisfiable, there is an R apt for DSDL3 such that $\mathsf{R} \models_{\overline{\mathrm{H}}} \gamma$ for every $\gamma \in \Gamma_f$, in which case $\mathsf{R} \models_{\overline{\mathrm{H}}} \beta \wedge \neg\beta$, which is impossible. Hence, Γ is satisfiable in DSDL3 models. ■

These are the results to be established in this section. Next we consider the weaker systems DSDL1, DSDL2 and DSDL2.5. In passing, however, we will also return to DDL-d and DSDL3 to prove them equivalent by a somewhat different method. That is for application later, in §6.2, where we find all the logics to be decidable.

5 DDL-a = DSDL1, DDL-b = DSDL2, and DDL-c = DSDL2.5

This section demonstrates that DDL-a is equivalent to DSDL1, DDL-b to DSDL2 and DDL-c to DSDL2.5. Along the way, as alluded above, we also return to the equivalence of DDL-d and DSDL3. To establish these results is tantamount to proving the soundness and completeness of the axiomatic systems with respect to the classes of Hanssonian models/relations R for the corresponding semantical system. We already have their soundness in Theorem 1.

For completeness, the demonstrations for the three weaker systems will be more difficult than that for DDL-d in the preceding section. They will proceed in two major stages. Stage 1 establishes the systems' completeness in terms of P-models. There we will find that for any α that is not a theorem of the logic there is an appropriate P-model that falsifies α. Unfortunately, the models we find are redundant, and we need irredundant models to match Hansson's definitions. To get around that, we will work at first not exactly with DDL-a, -b, and -c, but rather with their finite counterparts and finite, though redundant, P-models for them. From those, however, in Stage 2, we can derive Hanssonian relations R that also falsify α and are appropriate for DSDL1, DSDL2 and

DSDL2.5. With that, completeness for the full systems in terms of relations R follows. We apply similar procedures to DDL-d and its finite counterparts.

By 'finite counterparts' of these systems, I mean their analogs cast in a finite language. By 'finite language' I just mean a language with finitely many atoms. Section 2.1 specified the full, infinite languages $\mathcal{L}_{\mathrm{BL}}$ and $\mathcal{L}_{\mathrm{DL}^a}$. We now wish to speak of their finite sublanguages. For any finite $n \geq 1$, let $\mathcal{L}^n_{\mathrm{BL}}$ be a standard propositional language, just like $\mathcal{L}_{\mathrm{BL}}$, except that $\mathcal{L}^n_{\mathrm{BL}}$ has exactly n many atoms. From those, more complex formulas are formed as usual. We may suppose that if $n \leq m$, then $\mathcal{L}^n_{\mathrm{BL}} \subseteq \mathcal{L}^m_{\mathrm{BL}}$; also $\mathcal{L}^n_{\mathrm{BL}} \subset \mathcal{L}_{\mathrm{BL}}$.

$\mathcal{L}^n_{\mathrm{DL}^a}$ is simply the dyadic deontic language based on $\mathcal{L}^n_{\mathrm{BL}}$ as the full $\mathcal{L}_{\mathrm{DL}^a}$ is based on $\mathcal{L}_{\mathrm{BL}}$, with $O(B/A)$ well-formed when $A, B \in \mathcal{L}^n_{\mathrm{BL}}$. $\mathcal{L}^n_{\mathrm{DL}^a} \subset \mathcal{L}_{\mathrm{DL}^a}$. If a formula $\alpha \in \mathcal{L}^n_{\mathrm{DL}^a}$, but $\alpha \notin \mathcal{L}^{n-1}_{\mathrm{DL}^a}$, let us say that α is from level n, and write $\lambda(\alpha) = n$. Plainly, for every $\alpha \in \mathcal{L}_{\mathrm{DL}^a}$, there is a finite $n \geq 1$ such that $\lambda(\alpha) = n$.

If $\mathbf{L}$ is one of the logics DDL-a, -b, -c or -d, so, for finite $n \geq 1$, the finite logic $\mathbf{L}^n$ is a set of formulas from $\mathcal{L}^n_{\mathrm{DL}^a}$. These are determined axiomatically by all instances of the schemas of $\mathbf{L}$ that are formulas of $\mathcal{L}^n_{\mathrm{DL}^a}$, or that follow from those by the rules of inference restricted to formulas of $\mathcal{L}^n_{\mathrm{DL}^a}$. This should be clear enough. Notice we do not presume that $\mathbf{L}^n = \mathbf{L} \cap \mathcal{L}^n_{\mathrm{DL}^a}$. While that is true, to demonstrate it calls on results yet to be established; see Corollary 89 in §6.2 below. For now, $\mathbf{L}^n$ always refers to the axiomatic system given by the limitation of the postulates of $\mathbf{L}$ to $\mathcal{L}^n_{\mathrm{DL}^a}$. DDL^n-a is the counterpart of DDL-a in finite $\mathcal{L}^n_{\mathrm{DL}^a}$; similarly DDL^n-b, DDL^n-c, and DDL^n-d.

A P-model defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ is understood to be just that, defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ and not for any other, richer language. That is, if $M = \langle W, \leq, v\rangle$ is defined for $\mathcal{L}^n_{\mathrm{DL}^a}$, then $v(p)$ is defined for every atom $p \in \mathcal{L}^n_{\mathrm{BL}}$, and for no others. W may still be any nonempty set and $\leq \subseteq W \times W$. In light of this, a P-model M defined for $\mathcal{L}^n_{\mathrm{DL}^a}$, if replete, is understood to be replete for $\mathcal{L}^n_{\mathrm{BL}}$, not the whole of $\mathcal{L}_{\mathrm{BL}}$, i.e.,

- A P-model $M = \langle W, \leq, v\rangle$ defined for $\mathcal{L}^n_{\mathrm{DL}^a}$, is replete for $\mathcal{L}^n_{\mathrm{BL}}$ just in case, for all $A \in \mathcal{L}^n_{\mathrm{BL}}$, if $|A| \neq \varnothing$ then $|A|_M \neq \varnothing$.

Similarly, a P-model $M = \langle W, \leq, v\rangle$ defined for $\mathcal{L}^n_{\mathrm{DL}^a}$, is, if said to be limited or stoppered, understood to be limited or stoppered with respect to $\mathcal{L}^n_{\mathrm{BL}}$, i.e.,

- $M = \langle W, \leq, v\rangle$ defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ is limited for $\mathcal{L}^n_{\mathrm{BL}}$ just in case, for every $A \in \mathcal{L}^n_{\mathrm{BL}}$, if $|A|_M \neq \varnothing$, then $\mathrm{Max}_{\leq}(|A|_M) \neq \varnothing$;

- $M = \langle W, \leq, v\rangle$ defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ is stoppered for $\mathcal{L}^n_{\mathrm{BL}}$ just in case, for every $A \in \mathcal{L}^n_{\mathrm{BL}}$, and every $w \in W$, if $w \in |A|_M$, then $w \in \mathrm{Max}_{\leq}(|A|_M)$ or there is a $w' \in \mathrm{Max}_{\leq}(|A|_M)$ such that $w' \prec w$.

We now take the first small step on our journey. This lemma applies to all of the systems $\mathbf{L}$ = DDL-a, -b, -c, and -d, and their finite counterparts $\mathbf{L}^n$.

Lemma 14 *For any finite $n \geq 1$ and any $\alpha \in \mathcal{L}_{\mathrm{DL}^a}$, if $\lambda(\alpha) = n$, then if $\vdash \alpha$ in $\mathbf{L}^n$, then $\vdash \alpha$ in $\mathbf{L}$, with $\mathbf{L}^n$ the finite counterpart of $\mathbf{L}$ in the language $\mathcal{L}^n_{\mathrm{DL}^a}$ over $\mathcal{L}^n_{\mathrm{BL}}$.*

Proof. Obvious, since any derivation of α in $\mathbf{L}^n$ will be a derivation of α in $\mathbf{L}$, given that $\mathcal{L}^n_{\mathrm{DL}^a} \subset \mathcal{L}_{\mathrm{DL}^a}$. ■

Now the work begins.

5.1 Stage 1

This stage of the argument demonstrates the soundness and completeness of each system $\mathbf{L}^n$ in the framework of finite P-models of the appropriate kinds. As usual, soundness is routine, and so, for the most part, left to the reader to verify.

Theorem 15 *For each finite $n \geq 1$, (i) DDL^n-a is sound with respect to the class of P-models defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ that are replete for $\mathcal{L}^n_{\mathrm{BL}}$ and whose relation $\leq$ is reflexive; (ii) DDL^n-b is sound with respect to the class of P-models defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ that are replete for $\mathcal{L}^n_{\mathrm{BL}}$ and whose relation $\leq$ is reflexive and limited for $\mathcal{L}^n_{\mathrm{BL}}$; (iii) DDL^n-c is sound with respect to the class of P-models defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ that are replete for $\mathcal{L}^n_{\mathrm{BL}}$ and whose relation $\leq$ is reflexive and stoppered for $\mathcal{L}^n_{\mathrm{BL}}$. (iv) DDL^n-d is sound with respect to the class of P-models defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ that are replete for $\mathcal{L}^n_{\mathrm{BL}}$ and whose relation $\leq$ is reflexive, transitive and total, as well as limited for $\mathcal{L}^n_{\mathrm{BL}}$.*

Proof. As with Theorem 1, this is proved by demonstrating that all the systems' axioms are valid with respect to the appropriate classes of P-models and that the rules preserve validity. The demonstrations mimic those of Theorem 1, except that we note that, in the framework of P-models, to validate (RP) for DDL^n-b, DDL^n-c and DDL^n-d requires repletion as well as being limited; repletion is idle for DDL^n-a. Thus, for the validity of (RP), suppose $A \in \mathcal{L}^n_{\mathrm{BL}}$ is BL-consistent, i.e., $|A| \neq \varnothing$. Then, for any M that is replete for $\mathcal{L}^n_{\mathrm{BL}}$, $|A|_M \neq \varnothing$. If M is limited or stoppered for $\mathcal{L}^n_{\mathrm{BL}}$ as well as replete, then $\mathrm{Max}_{\mathsf{R}}(|A|_M) \neq \varnothing$. From that, $M \models_{\mathrm{P}} P(\top/A)$. Hence, $\Vdash_{\mathrm{P}} P(\top/A)$. ■

For later reference, we note that adding a condition of finiteness would not affect the validity of any of the postulates of DDL^n; hence:

Corollary 16 *For each DDL^n system, DDL^n is sound with respect to the class of finite P-models appropriate to DDL^n.*

We turn now to completeness. For this, we give one sort of demonstration for DDL^n-a and DDL^n-b and another for DDL^n-c, because the first construction is not conducive to stoppered relations, while the second requires (CautMono) of DDL^n-c. For both methods, however, the arguments are chiefly technical; there is no natural motivation or informal explanation for the various devices used along the way, except that they accomplish the desired results. For DDL^n-d the proof will echo that of §4.

5.1.1 Finite DDL^n-a and DDL^n-b

Assume finite $n \geq 1$ is given. We now demonstrate that the finite-based DDL^n-a and DDL^n-b are complete with respect to the class of P-models defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ that are (i) finite and (ii) replete for $\mathcal{L}^n_{\mathrm{BL}}$ and, for DDL^n-b, (iii) limited for $\mathcal{L}^n_{\mathrm{BL}}$.[5]

[5] This demonstration draws on methods used by Parent [12] to prove completeness for Åqvist's systems **E** and **F**, which are similar to DDL-a and DDL-b, though also significantly different. Parent cites Schlechta [15] as a source of some of his ideas. I have drawn too from Schlechta's [14], though it has been necessary to adjust his methods to suit the present systems. Indeed, the present demonstration diverges considerably from both Schlechta's and from Parent's; I need not describe those differences here, however.

To begin, let w^n be a maximal DDL^n-a or DDL^n-b consistent set of formulas of $\mathcal{L}^n_{\mathrm{DL}^a}$. Much as before, let $\Delta_A = \{B \in \mathcal{L}^n_{\mathrm{BL}} : O(B/A) \in \mathsf{w}^n\}$. Lemma 4 for DDL-d continues to hold here, with w^n in place of w.

Let W^{BL^n} be the set of all maximal BL-consistent sets of formulas of $\mathcal{L}^n_{\mathrm{BL}}$. Let W^t be a triplication of W^{BL^n}; i.e., let

- $W^1 = \{\langle x, 1\rangle : x \in W^{\mathrm{BL}^n}\}$,
- $W^2 = \{\langle x, 2\rangle : x \in W^{\mathrm{BL}^n}\}$,
- $W^3 = \{\langle x, 3\rangle : x \in W^{\mathrm{BL}^n}\}$,
- $W^t = W^1 \cup W^2 \cup W^3$.

This multiplication is required for Lemma 21 below, which is key to subsequent results.

Henceforth, I will use letters, x, y, z, etc., as variables for members of W^{BL^n}, and letters a, b, c, etc., as variables for members of W^t. When $a = \langle x, i\rangle$, for $i \in \{1, 2, 3\}$, I will write a' and a'' for its two images, $\langle x, j\rangle$ and $\langle x, k\rangle$ for $j, k \in \{1, 2, 3\}$ where $i \neq j \neq k \neq i$. Generally, it will not matter which is which, only that $a \neq a' \neq a'' \neq a$.

For all $A \in \mathcal{L}^n_{\mathrm{BL}}$, let

- $[A] = \{\langle x, i\rangle : x \in W^{\mathrm{BL}^n}$ and $i \in \{1, 2, 3\}$ and $A \in x\}$.

Thus, for $a = \langle x, i\rangle$, $a \in [A]$ just in case $A \in x$. Given the multiplication inherent in W^t, it is apparent that each $[A]$ contains the images of its members.

Lemma 17 *For all $a \in W^t$ and all $A \in \mathcal{L}^n_{\mathrm{BL}}$, $a \in [A]$ iff $a' \in [A]$ iff $a'' \in [A]$, a' and a'' being the images of a in W^t.*

Proof. Obvious. ■

For points $a \in W^t$, and formulas $A \in \mathcal{L}^n_{\mathrm{BL}}$, let us say, much as before,

- a is w^n-normal for A iff, for all B such that $O(B/A) \in \mathsf{w}^n$, $a \in [B]$.

Thus, for $a = \langle x, i\rangle$, a is w^n-normal for A just in case $\Delta_A \subseteq x$. Henceforth, as before, we drop the prefix, and write simply 'a is normal'.

Lemma 18 *For all $A, B \in \mathcal{L}^n_{\mathrm{BL}}$, if $[A] = [B]$ then (i) $\vdash A \leftrightarrow B$, and (ii) a is normal for A iff a is normal for B.*

Proof. Suppose $[A] = [B]$. For (i), if $\nvdash A \leftrightarrow B$ then $\neg(A \leftrightarrow B)$ is consistent. So there is an $x \in W^{\mathrm{BL}^n}$ containing either A and $\neg B$ or else $\neg A$ and B. Consider the first case; the second is similar. Let $a = \langle x, 1\rangle$. $a \in [A]$; so $a \in [B]$. Then $B \in x$, contrary to its consistency. Hence, $\vdash A \leftrightarrow B$. (ii) follows from (i), since if $\vdash A \leftrightarrow B$, $\Delta_A = \Delta_B$ by (LLE). So, for $a = \langle x, i\rangle$, a is normal for A iff $\Delta_A \subseteq x$ iff $\Delta_B \subseteq x$ iff a is normal for B. ■

For all $a \in W^t$, let

- $\Upsilon a = \{X \subseteq W^t$: there is an $A \in \mathcal{L}^n_{\mathrm{BL}}$ such that $X = [A]$ and $a \in [A]$ and a is not normal for $A\}$.

Lemma 19 *Υa is well-defined.*

Proof. Obvious, by the quantification built into the definition. We also note, however, that if $[A] = [B]$, then a is normal for A iff a is normal for B, Lemma 18(ii); since also $a \in [A]$ iff $a \in [B]$, then, under the definition, $[A] \in \Upsilon a$ iff $[B] \in \Upsilon a$. ■

In general, given a set, $\mathcal{X}$, of nonempty sets, X, let us say:

- χ is a *sampler set* over $\mathcal{X}$ just in case both (i) for every $X \in \mathcal{X}$, there is an $x \in X$ such that $x \in \chi$, and also (ii) for every $x \in \chi$, there is an $X \in \mathcal{X}$ such that $x \in X$.

Thus, predictably, a sampler set χ over $\mathcal{X}$ is composed of samples from all the $X \in \mathcal{X}$.

Lemma 20 *For any set, $\mathcal{X}$, of nonempty sets there is a sampler set χ over $\mathcal{X}$.*

Proof. Given $\mathcal{X}$, let $\chi = \bigcup \mathcal{X}$. χ is a sampler set over $\mathcal{X}$, by the definition. ■

In case $\mathcal{X}$ is empty, then the only sampler set over $\mathcal{X}$ will be $\varnothing$, but it still counts.

Obviously, for $a \in W^t$, Υa is a set of nonempty subsets, though it might be empty. For all $a \in W^t$, let

- $\Phi a = \{\chi : \chi$ is a sampler set over Υa and $a \notin \chi\}$.

Lemma 21 *(i) For all $a \in W^t$, $\Phi a \neq \varnothing$. (ii) For every $a, b \in W^t$, there is a $\chi \in \Phi b$ such that $a \notin \chi$.*

Proof. (i) follows directly from (ii). For that, consider $\chi = \bigcup \Upsilon b - \{a, b\}$. We show that χ is a sampler set over Υb. First, consider any $X \in \Upsilon b$, so that there is an A such that $X = [A]$ and $b \in [A]$. By Lemma 17, $b' \in [A]$ and $b'' \in [A]$, i.e., $b' \in X$ and $b'' \in X$, for b', b'' the two images of b. In case $a = b'$, then $b'' \in \chi$; in case $a \neq b'$, then both $b' \in \chi$ and $b'' \in \chi$. In either case, (a), for any $X \in \Upsilon b$, there is a $c \in X$ such that $c \in \chi$, as required for a sampler set. Next, consider any $c \in \chi$. Then $c \in \bigcup \Upsilon b$, so that (b), for any $c \in \chi$, there is an $X \in \Upsilon b$ such that $c \in X$, as also required for a sampler set. By (a) and (b) together, χ is a sampler set over Υb. Since $b \notin \chi$, $\chi \in \Phi b$. Obviously, $a \notin \chi$. Hence, there is a $\chi \in \Phi b$ such that $a \notin \chi$. ■

We are now in a position to define our intended model. Let $M = \langle W, \leq, v\rangle$, with

- $W = \{\langle a, \chi\rangle : a \in W^t$ and $\chi \in \Phi a\}$,
- for $\langle a, \chi\rangle, \langle b, \chi'\rangle \in W$, $\langle a, \chi\rangle \leq \langle b, \chi'\rangle$ iff either (i) $\langle a, \chi\rangle = \langle b, \chi'\rangle$ or (ii) $a \in \chi'$,
- $v(p) = \{\langle a, \chi\rangle \in W : a \in [p]\}$, for each atom $p \in \mathcal{L}^n_{\mathrm{BL}}$.

Lemma 22 *M is a P-model defined for $\mathcal{L}^n_{\mathrm{DL}^a}$, and $\leq$ is reflexive.*

Proof. This should be obvious, or nearly so. Given Lemma 21(i), for every $a \in W^t$, there is a $\chi \in \Phi a$. Since there are $a \in W^t$, there are thus $\langle a, \chi\rangle \in W$. Hence, $W \neq \varnothing$. Clearly, $\leq \subseteq W \times W$, and v is well-defined for all and only atoms $p \in \mathcal{L}^n_{\mathrm{BL}}$. Hence M is a P-model defined for $\mathcal{L}^n_{\mathrm{DL}^a}$. That $\leq$ is reflexive is given by clause (i) of its definition. ■

It remains to show that M does the work it is supposed to do. With points $\langle a, \chi\rangle$, $\langle b, \chi'\rangle$, etc. now in play, let us write '$|[A]|$' to signify the set of points in W whose left member belongs to $[A]$. I.e.,

- $|[A]| = \{\langle a, \chi\rangle \in W : a \in [A]\}$.

Lemma 23 *For all $a \in W^t$ and all $A \in \mathcal{L}^n_{\mathrm{BL}}$, a is normal for A iff there is a $\chi \in \Phi a$ such that $\langle a, \chi\rangle \in \mathrm{Max}_{\preceq}(|[A]|)$.*

Proof. L → R: Suppose a is normal for A. By virtue of (Reflex), $a \in [A]$. Consider the set $\chi = \bigcup \Upsilon a - [A]$. We show first (1) $\chi \in \Phi a$ and then (2) $\langle a, \chi\rangle \in \mathrm{Max}_{\preceq}(|[A]|)$. For (1), we show that χ is a sampler set over Υa. First, consider any $X \in \Upsilon a$, so that there is a B such that $X = [B]$ and $a \in [B]$ and a is not normal for B. We show first that $[B] - [A] \neq \varnothing$. Suppose, for *reductio*, $[B] - [A] = \varnothing$. Then $[B] \subseteq [A]$, whence $[A] \cap [B] = [B]$. It is not hard to show, and so left to the reader, that $[A] \cap [B] = [A \wedge B]$. That being so, $[B] = [A \wedge B]$. Thus, a is not normal for $A \wedge B$, by Lemma 18(ii). On the other hand, consider any C such that $O(C/A \wedge B) \in \mathsf{w}^n$. By principle (S), proved in Lemma 5(1), $O(B \to C/A) \in \mathsf{w}^n$. Since a is normal for A, $a \in [B \to C]$, so that, if $a = \langle x, i\rangle$, $B \to C \in x$. Given $a \in [B]$, then $B \in x$; it follows that $C \in x$; hence, $a \in [C]$. That suffices for a's being normal for $A \wedge B$, a contradiction. Therefore, $[B] - [A] \neq \varnothing$. Thus, there is a $c \in [B]$, i.e., $c \in X$, such that $c \notin [A]$. Then $c \in \bigcup \Upsilon a - [A]$, i.e., $c \in \chi$. Thus (i), for any $X \in \Upsilon a$, there is a $c \in X$ such that $c \in \chi$. Next, consider any $c \in \chi$. So, $c \in \bigcup \Upsilon a$, which is to say, there is an $X \in \Upsilon a$ such that $c \in X$. Thus (ii), for any $c \in \chi$, there is an $X \in \Upsilon a$ such that $c \in X$. By (i) and (ii), χ is a sampler set over Υa. Furthermore, $a \notin \chi$, for suppose, for *reductio*, $a \in \chi$. Then $a \in \bigcup \Upsilon a - [A]$; so $a \notin [A]$, a contradiction. Therefore, $a \notin \chi$. Since χ is a sampler set over Υa and $a \notin \chi$, (1) $\chi \in \Phi a$.

Next we show that $\langle a, \chi\rangle \in \mathrm{Max}_{\preceq}(|[A]|)$. Since (1) $\chi \in \Phi a$, $\langle a, \chi\rangle \in W$, and since $a \in [A]$, $\langle a, \chi\rangle \in |[A]|$. To see that it is maximal in $|[A]|$, suppose, for *reductio*, there were some $\langle b, \chi'\rangle \in |[A]|$ such that $\langle b, \chi'\rangle \prec \langle a, \chi\rangle$. Obviously, $\langle b, \chi'\rangle \neq \langle a, \chi\rangle$. Hence, by definition of $\preceq$, and so of $\prec$, $b \in \chi$. Since $\langle b, \chi'\rangle \in |[A]|$, $b \in [A]$. Since $b \in \chi$, $b \in \bigcup \Upsilon a$ and $b \notin [A]$, a contradiction. Hence there is no such $\langle b, \chi'\rangle$, and so (2) $\langle a, \chi\rangle \in \mathrm{Max}_{\preceq}(|[A]|)$.

R → L: Suppose some $\chi \in \Phi a$ such that $\langle a, \chi\rangle \in \mathrm{Max}_{\preceq}(|[A]|)$. Since $\langle a, \chi\rangle \in |[A]|$, $a \in [A]$. Suppose, for *reductio*, a is not normal for A. Then $[A] \in \Upsilon a$. By definition of Φa, χ is a sampler set over Υa, and so there must be some $b \in [A]$ and $b \in \chi$. Since $\chi \in \Phi a$, $a \notin \chi$, and since $b \in \chi$, $a \neq b$. By Lemma 21(ii), there is a $\chi' \in \Phi b$ such that $a \notin \chi'$. $\langle b, \chi'\rangle \in W$. Also, $\langle b, \chi'\rangle \in |[A]|$. Since $b \in \chi$, $\langle b, \chi'\rangle \preceq \langle a, \chi\rangle$. Since $\langle a, \chi\rangle \in \mathrm{Max}_{\preceq}(|[A]|)$, it follows that $\langle a, \chi\rangle \preceq \langle b, \chi'\rangle$. And since $a \neq b$, $\langle a, \chi\rangle \neq \langle b, \chi'\rangle$; hence, it must be that $a \in \chi'$, a contradiction. Therefore, a is normal for A. ■

Lemma 24 *For all $A \in \mathcal{L}^n_{\mathrm{BL}}$, $|[A]| = |A|_M$.*

Proof. By an easy induction on A, easy enough to leave to the reader. ■

These preliminaries enable our key lemma:

Lemma 25 *For all $\alpha \in \mathcal{L}^n_{\mathrm{DL}^a}$, $\alpha \in \mathsf{w}^n$ iff $M \models_{\mathrm{P}} \alpha$.*

Proof. By induction on α. We show the basis case, where $\alpha = O(B/A)$. The induction to more complex formulas is routine, and left to the reader.

L → R: Suppose $O(B/A) \in \mathsf{w}^n$. To show that $M \models_{\mathrm{P}} O(B/A)$, i.e., that $\mathrm{Max}_{\preceq}(|A|_M) \subseteq |B|_M$, suppose some $\langle a, \chi\rangle \in \mathrm{Max}_{\preceq}(|A|_M)$. By Lemma 24, $\langle a, \chi\rangle \in \mathrm{Max}_{\preceq}(|[A]|)$. Since

$\langle a,\chi\rangle \in W$, $\chi \in \Phi a$. Hence there is a χ such that $\chi \in \Phi a$ and $\langle a,\chi\rangle \in \mathrm{Max}_{\leq}(|[A]|)$. By Lemma 23, a is normal for A. Hence, $a \in [B]$. Then $\langle a,\chi\rangle \in |[B]|$, and so $\langle a,\chi\rangle \in |B|_M$, by Lemma 24. That suffices for $\mathrm{Max}_{\leq}(|A|_M) \subseteq |B|_M$, and so for $M \models_{\mathrm{P}} O(B/A)$.

R → L: Suppose $M \models_{\mathrm{P}} O(B/A)$, so that $\mathrm{Max}_{\leq}(|A|_M) \subseteq |B|_M$. Since $O(B/A) \in \mathcal{L}^n_{\mathrm{DL}^a}$, $A, B \in \mathcal{L}^n_{\mathrm{BL}}$. By Lemma 24, $|[A]| = |A|_M$ and $|[B]| = |B|_M$. Hence, $\mathrm{Max}_{\leq}(|[A]|) \subseteq |[B]|$. Suppose, for *reductio*, $O(B/A) \notin \mathrm{w}^n$. Then, by Lemma 4(ii), $\Delta_A \cup \{\neg B\}$ is BL-consistent. Hence there is an $x \in W^{\mathrm{BL}^n}$ such that $\Delta_A \cup \{\neg B\} \subseteq x$. Let $a = \langle x, 1\rangle$. Since $\Delta_A \subseteq x$, a is normal for A. By Lemma 23, there is a $\chi \in \Phi a$ such that $\langle a,\chi\rangle \in \mathrm{Max}_{\leq}(|[A]|)$. Thus, $\langle a,\chi\rangle \in |[B]|$, so that $a \in [B]$, which means that $B \in x$. But also $\neg B \in x$, contrary to its consistency. Hence it must be that $O(B/A) \in \mathrm{w}^n$. ■

Lemma 26 *For finite $\mathcal{L}^n_{\mathrm{BL}}$, M is (i) finite and (ii) replete for $\mathcal{L}^n_{\mathrm{BL}}$.*

Proof. (i) That M is finite follows from $\mathcal{L}^n_{\mathrm{BL}}$ being finite. Thus, there are only finitely many $x \in W^{\mathrm{BL}^n}$, so only finitely many $a \in W^t$. Further, for any $a \in W^t$, since for any $\chi \in \Phi a$, $\chi \subseteq W^t$, there can be only finitely many such χ's. As a result, there are only finitely many points $\langle a,\chi\rangle \in W$.

For (ii), for $A \in \mathcal{L}^n_{\mathrm{BL}}$, suppose $|A| \neq \varnothing$. Thus A is BL-consistent, and so there is an $x \in W^{\mathrm{BL}^n}$ such that $A \in x$. Let $a = \langle x, 1\rangle$. By Lemma 21(i), there is a $\chi \in \Phi a$. For such a one, $\langle a,\chi\rangle \in W$. Since $A \in x$, $a \in [A]$, so $\langle a,\chi\rangle \in |[A]|$. By Lemma 24, $\langle a,\chi\rangle \in |A|_M$; hence, $|A|_M \neq \varnothing$, as required for repletion. ■

Lemma 27 *If w^n is an extension of DDL^n-b, then M is limited for $\mathcal{L}^n_{\mathrm{BL}}$.*

Proof. This follows from (RP) being in DDL^n-b. Suppose $A \in \mathcal{L}^n_{\mathrm{BL}}$ such that $|A|_M \neq \varnothing$. By Lemma 24, $|[A]| \neq \varnothing$. Suppose then $\langle a,\chi\rangle \in |[A]|$, so that $a \in [A]$, and suppose $a = \langle x, i\rangle$ for some $x \in W^{\mathrm{BL}^n}$ and $i \in \{1, 2, 3\}$. Since $a \in [A]$, $A \in x$. Thus A is BL-consistent. By (RP), $\vdash P(\top/A)$ in DDL-b; hence, $P(\top/A) \in \mathrm{w}^n$. By Lemma 4(i), Δ_A is consistent. Hence there is a $y \in W^{\mathrm{BL}^n}$ such that $\Delta_A \subseteq y$. Let $b = \langle y, 1\rangle$. $b \in W^t$. Since b is normal for A, there is a $\chi' \in \Phi b$ such that $\langle b,\chi'\rangle \in \mathrm{Max}_{\leq}(|[A]|)$, by Lemma 23. So by Lemma 24, $\langle b,\chi'\rangle \in \mathrm{Max}_{\leq}(|A|_M)$, and thus $\mathrm{Max}_{\leq}(|A|_M) \neq \varnothing$, as required for M to be limited for $\mathcal{L}^n_{\mathrm{BL}}$. ■

From these completeness follows quickly in the usual way.

Theorem 28 *For all finite $n \geq 1$, (i) DDL^n-a is weakly complete with respect to the class of all P-models defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ that are finite and replete for $\mathcal{L}^n_{\mathrm{BL}}$ and whose relation $\leq$ is reflexive, i.e., for any $\alpha \in \mathcal{L}^n_{\mathrm{DL}^a}$, if $\Vdash_{\mathrm{P}} \alpha$ for that class, then $\vdash \alpha$ in DDL^n-a.*

(ii) DDL^n-b is weakly complete with respect to the class of all P-models defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ that are finite and replete for $\mathcal{L}^n_{\mathrm{BL}}$ and whose relation $\leq$ is reflexive and limited for $\mathcal{L}^n_{\mathrm{BL}}$, i.e., for any $\alpha \in \mathcal{L}^n_{\mathrm{DL}^a}$, if $\Vdash_{\mathrm{P}} \alpha$ for that class, then $\vdash \alpha$ in DDL^n-b.

Proof. Given $n \geq 1$ and $\alpha \in \mathcal{L}^n_{\mathrm{BL}}$, for (i) suppose $\nvdash \alpha$ in DDL^n-a. Then $\neg\alpha$ is consistent for DDL^n-a. Let w^n be a maximal DDL^n-a consistent set of $\mathcal{L}^n_{\mathrm{DL}^a}$ formulas such that $\neg\alpha \in \mathrm{w}^n$. By the usual arguments, we know there is one. Let M be defined from w^n as described above. By Lemma 22, M is a P-model defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ and $\leq$ is reflexive. By Lemma 26, M is finite and replete for $\mathcal{L}^n_{\mathrm{BL}}$. By Lemma 25, $M \models_{\mathrm{P}} \neg\alpha$; hence, $M \not\models_{\mathrm{P}} \alpha$. It follows that $\nVdash_{\mathrm{P}} \alpha$ for the class specified. Hence, if $\Vdash_{\mathrm{P}} \alpha$ for that class, then $\vdash \alpha$ in DDL^n-a. The argument for (ii) is the same vis-à-vis DDL^n-b, with the addition of Lemma 27 that assures that $\preceq$ of M is limited for $\mathcal{L}^n_{\mathrm{BL}}$. ■

While this theorem emphasizes completeness with respect to finite models, that is only because that is what will be applied in Stage 2 of the overall argument. The demonstration could, however, easily be adapted to establish the completeness of the full systems, DDL-a and DDL-b in the infinite language $\mathcal{L}_{\mathrm{DL}^a}$ in terms of infinite P-models. Moreover, while this theorem only asserts weak completeness for these systems, the argument is also readily adapted to establish the strong completeness of the full systems, for appropriate classes of P-models.

Finitude enters the picture in another way as well. By Theorem 15 and its Corollary 16 and Theorem 28, for each finite $n \geq 1$, each DDL^n-a and DDL^n-b is characterized by a class of finite models, which is to say, it has the finite model property.

Corollary 29 *For each finite $n \geq 1$, each DDL^n-a and DDL^n-b has the finite model property in terms of appropriate P-models defined for $\mathcal{L}^n_{\mathrm{DL}^a}$.*

Hence, importantly, by the usual arguments, because DDL^n-a and DDL^n-b are finitely axiomatizable:

Corollary 30 *For each finite $n \geq 1$, each DDL^n-a and DDL^n-b is decidable.*[6]

That completes this stage for DDL^n-a and DDL^n-b. We turn next to DDL^n-c.

5.1.2 Finite DDL^n-c

For completeness for finite DDL^n-c, we develop a rather different model than we saw in the preceding.[7] To construct this model, much as before, let w^n be a maximal DDL^n-c consistent set of $\mathcal{L}^n_{\mathrm{DL}^a}$ formulas. W^{BL^n} is the set of maximal BL-consistent sets of $\mathcal{L}^n_{\mathrm{BL}}$ formulas. For all $A \in \mathcal{L}^n_{\mathrm{BL}}$, let

- $[A] = \{x \in W^{\mathrm{BL}^n} : A \in x\}$.
- $\Delta_A = \{B : O(B/A) \in \mathsf{w}^n\}$.

Also much as before,

- $x \in W^{\mathrm{BL}^n}$ is normal for A iff $\Delta_A \subseteq x$.

Analogous to Lemma 18 for DDL^n-a, here are some useful little tools.

Lemma 31 *(i) If $[A] = [B]$, then $\vdash A \leftrightarrow B$; (ii) if $[A] = [B]$, then for any $x \in W^{\mathrm{BL}^n}$, x is normal for A iff x is normal for B; (iii) if $[A] = [C]$ and $[B] = [D]$ then $O(A/A \vee B) \in \mathsf{w}^n$ iff $O(C/C \vee D) \in \mathsf{w}^n$.*

[6] On a logic's being decidable if it has the finite model property and is finitely axiomatizable, see standard sources on modal logic, such as Chellas [1], §2.8, Cresswell [2], §7.1.4, or Hughes and Cresswell [6], pp. 152–3.

[7] The demonstration here draws on the proof of the representation theorem for the logic **P** for preferential reasoning given by Kraus, Lehmann and Magidor, [7] §5, Theorem 5.18, though streamlined now and adapted to suit the needs of the present framework, and my own style of doing things.

Proof. (i) and (ii) are easy and left to the reader; apply the argument for Lemma 18 for DDL^n-a. (iii) follows from (i). Thus, if $[A] = [C]$ and $[B] = [D]$ then $\vdash A \leftrightarrow C$ and $\vdash B \leftrightarrow D$, whence $\vdash (A \vee B) \leftrightarrow (C \vee D)$. Then, $O(A/A \vee B) \in \mathsf{w}^n$ iff $O(C/C \vee D) \in \mathsf{w}^n$, by (LLE) and (RW). ■

We now define the model, $M = \langle W, \preceq, v\rangle$, where

- $W = \{\langle x, X\rangle : x \in W^{\mathrm{BL}^n}$ and there is an $A \in \mathcal{L}^n_{\mathrm{BL}}$ such that $X = [A]$ and x is normal for $A\}$,
- for $\langle x, X\rangle, \langle y, Y\rangle \in W$, $\langle x, X\rangle \preceq \langle y, Y\rangle$ iff either (i) $\langle x, X\rangle = \langle y, Y\rangle$, or (ii) for all $A, B \in \mathcal{L}^n_{\mathrm{BL}}$, if $X = [A]$ and $Y = [B]$, then $O(A/A \vee B) \in \mathsf{w}^n$ and $x \notin [B]$,
- $v(p) = \{\langle x, X\rangle \in W : p \in x\}$.

Lemma 32 *M is a P-model defined for* $\mathcal{L}^n_{\mathrm{DL}^a}$.

Proof. Since all components of M are well-defined, we need only show that $W \neq \varnothing$. For that, consider that $\top \in \mathcal{L}^n_{\mathrm{BL}}$ is consistent. Hence, by (RP) $\vdash P(\top/\top)$ in DDL^n-c, and then $P(\top/\top) \in \mathsf{w}^n$. By Lemma 4(i), $\Delta_\top$ is consistent, and so there is an $x \in W^{\mathrm{BL}^n}$ such that $\Delta_\top \subseteq x$. Thus x is normal for $\top$, and so $\langle x, [\top]\rangle \in W$, and $W \neq \varnothing$. ■

Lemma 33 $\preceq$ *is (i) reflexive, (ii) transitive.*

Proof. (i) Reflexivity is trivial by clause (i) of the definition of $\preceq$. (ii) For transitivity, suppose $\langle x, X\rangle \preceq \langle y, Y\rangle$ and $\langle y, Y\rangle \preceq \langle z, Z\rangle$. If either of those is by clause (i) of the definition of $\preceq$, it is trivial that $\langle x, X\rangle \preceq \langle z, Z\rangle$. So suppose both are by clause (ii). Thus, for all $A, B \in \mathcal{L}^n_{\mathrm{BL}}$, if $X = [A]$ and $Y = [B]$ then $O(A/A \vee B) \in \mathsf{w}^n$ and $x \notin [B]$, and for all $C, D \in \mathcal{L}^n_{\mathrm{BL}}$, if $Y = [C]$ and $Z = [D]$ then $O(C/C \vee D) \in \mathsf{w}^n$ and $y \notin [D]$. Consider any $A, D \in \mathcal{L}^n_{\mathrm{BL}}$ such that $X = [A]$ and $Z = [D]$. Since $\langle y, Y\rangle \in W$, there is a B such that $Y = [B]$ and y is normal for B. Then $O(A/A \vee B) \in \mathsf{w}^n$ and $x \notin [B]$, i.e., $B \notin x$, and also $O(B/B \vee D) \in \mathsf{w}^n$ and $y \notin [D]$. We show first $O(A/A \vee D) \in \mathsf{w}^n$. For that, given $O(A/A \vee B) \in \mathsf{w}^n$ and $O(B/B \vee D) \in \mathsf{w}^n$, $O(A \vee B/A \vee B) \in \mathsf{w}$ by (RW) or (Reflex) and $O(A \vee B/B \vee D) \in \mathsf{w}^n$ by (RW), whence (a) $O(A \vee B/A \vee B \vee D) \in \mathsf{w}^n$ by (OR) and (LLE). Since $O(A/A \vee B) \in \mathsf{w}^n$, $O(A/(A \vee B) \wedge (A \vee B \vee D)) \in \mathsf{w}^n$ by (LLE), whence (b) $O((A \vee B) \to A/A \vee B \vee D) \in \mathsf{w}^n$ by (S), Lemma 5(1). Then, with (a), $O((A \vee B) \wedge ((A \vee B) \to A)/A \vee B \vee D) \in \mathsf{w}^n$ by (AND), so that (c) $O(A/A \vee B \vee D) \in \mathsf{w}^n$ by (RW), and then (d) $O(A \vee D/A \vee B \vee D) \in \mathsf{w}^n$ by (RW) again. From (c) and (d), by (CautMono), $O(A/(A \vee D) \wedge (A \vee B \vee D)) \in \mathsf{w}^n$, and so, by (LLE), (e) $O(A/A \vee D) \in \mathsf{w}^n$, as desired. Next, we show $x \notin [D]$. Suppose, for *reductio*, $x \in [D]$, i.e., $D \in x$. Since $\langle x, X\rangle \in W$, there is an E such that $X = [E]$ and x is normal for E. Since thus $[E] = [A]$, x is normal for A, by Lemma 31(ii). Since $O(B/B \vee D) \in \mathsf{w}^n$, $O(B/(B \vee D) \wedge (A \vee B \vee D)) \in \mathsf{w}^n$ by (LLE), whence $O((B \vee D) \to B/A \vee B \vee D) \in \mathsf{w}^n$ by (S). Since $O(A/A \vee B \vee D) \in \mathsf{w}^n$, as (c) above, then $O((B \vee D) \to B/A \wedge (A \vee B \vee D)) \in \mathsf{w}^n$ by (CautMono), whence $O((B \vee D) \to B/A) \in \mathsf{w}^n$ by (LLE). Since x is normal for A, $(B \vee D) \to B \in x$. Thus, if $D \in x$, $B \vee D \in x$, so that $B \in x$, a contradiction. Therefore $D \notin x$ and so $x \notin [D]$. That and (e) $O(A/A \vee D) \in \mathsf{w}^n$ suffice for $\langle x, X\rangle \preceq \langle z, Z\rangle$. ■

For every $A \in \mathcal{L}^n_{\mathrm{BL}}$, let

- $|[A]| = \{\langle x, X\rangle \in W : x \in [A]\}$.

Lemma 34 *For all $x \in W^{\mathrm{BL}^n}$ and all $A \in \mathcal{L}^n_{\mathrm{BL}}$, (i) if x is normal for A then $\langle x, [A]\rangle \in \mathrm{Max}_{\preceq}(|[A]|)$; (ii) if, for any $X \subseteq W^{\mathrm{BL}^n}$, $\langle x, X\rangle \in \mathrm{Max}_{\preceq}(|[A]|)$, then x is normal for A.*

Proof. For (i), suppose x is normal for A. Given (Reflex), $x \in [A]$, so $\langle x, [A]\rangle \in |[A]|$. To show it to be maximal there, suppose, for *reductio*, there is some $\langle y, Y\rangle \in |[A]|$ such that $\langle y, Y\rangle \prec \langle x, [A]\rangle$. Obviously, $\langle y, Y\rangle \neq \langle x, [A]\rangle$; so, for all B and C, if $Y = [B]$ and $[A] = [C]$, then $O(B/B \vee C) \in \mathsf{w}^n$ and $y \notin [C]$. Hence, $y \notin [A]$. But then $\langle y, Y\rangle \notin |[A]|$, a contradiction. Hence, there is no such $\langle y, Y\rangle$, and so $\langle x, [A]\rangle \in \mathrm{Max}_{\preceq}(|[A]|)$.

For (ii), suppose, for some $X \subseteq W^{\mathrm{BL}^n}$, $\langle x, X\rangle \in \mathrm{Max}_{\preceq}(|[A]|)$. Since $\langle x, X\rangle \in W$, there is a C such that $X = [C]$ and x is normal for C. So $C \in x$ and $x \in [C]$. Also $x \in [A]$, and $A \in x$. To find x is normal for A, we show first that $O(C/A \vee C) \in \mathsf{w}^n$. For that, suppose, for *reductio*, $O(C/A \vee C) \notin \mathsf{w}^n$. Then $\Delta_{A\vee C} \cup \{\neg C\}$ is consistent, by Lemma 4(ii). Hence, there is a $y \in W^{\mathrm{BL}^n}$ such that $\Delta_{A\vee C} \cup \{\neg C\} \subseteq y$. Since y is normal for $A \vee C$, $\langle y, [A \vee C]\rangle \in W$. Since $\neg C \in y$, $C \notin y$, and $y \notin [C]$. Since $A \vee C \in y$ and $C \notin y$, then $A \in y$. So $y \in [A]$. Then $\langle y, [A \vee C]\rangle \in |[A]|$. We show $\langle y, [A \vee C]\rangle \preceq \langle x, X\rangle$. For that, consider any D and E such that $[A \vee C] = [D]$ and $X = [E] = [C]$. By (Reflex) and (LLE), $O(A \vee C/A \vee C \vee C) \in \mathsf{w}^n$. Hence $O(D/D \vee E) \in \mathsf{w}^n$, by Lemma 31(iii). Since $y \notin [C]$, $y \notin [E]$. That suffices for $\langle y, [A \vee C]\rangle \preceq \langle x, X\rangle$. Moreover, since $x \in [C]$, $x \neq y$, so $\langle x, X\rangle \neq \langle y, [A \vee C]\rangle$. Consequently, $\langle x, X\rangle \npreceq \langle y, [A \vee C]\rangle$, for, if $\langle x, X\rangle \preceq \langle y, [A \vee C]\rangle$ that must be by clause (ii), so that $x \notin [A \vee C]$, whereas $x \in [A \vee C]$ since $x \in [C]$. Since, thus, $\langle x, X\rangle \npreceq \langle y, [A \vee C]\rangle$, $\langle y, [A \vee C]\rangle \prec \langle x, X\rangle$. But then $\langle x, X\rangle \notin \mathrm{Max}_{\preceq}(|[A]|)$, a contradiction. Therefore, $O(C/A \vee C) \in \mathsf{w}^n$. Given that, it follows that x is normal for A, for consider any $D \in \Delta_A$, so that $O(D/A) \in \mathsf{w}^n$. Then $O(D/A \wedge (A \vee C)) \in \mathsf{w}^n$ by (LLE), and then $O(A \to D/A \vee C) \in \mathsf{w}^n$ by (S), Lemma 5(1). Since $O(C/A \vee C) \in \mathsf{w}^n$, $O(A \to D/C \wedge (A \vee C)) \in \mathsf{w}^n$ by (CautMono). Hence, $O(A \to D/C) \in \mathsf{w}^n$ by (LLE). From that, $A \to D \in x$ since x is normal for C. With $A \in x$, then $D \in x$, which suffices for $\Delta_A \subseteq x$ and so for x to be normal for A. ∎

We can now establish our key lemmas.

Lemma 35 *For all $A \in \mathcal{L}^n_{\mathrm{BL}}$, $|[A]| = |A|_M$.*

Proof. By an easy induction on A, left to the reader. ∎

Lemma 36 *For all $\alpha \in \mathcal{L}^n_{\mathrm{DL}^a}$, $\alpha \in \mathsf{w}^n$ iff $M \models_{\mathrm{P}} \alpha$.*

Proof. Proof is by induction on α, but we consider only the basis case where $\alpha = O(B/A)$, for some $A, B \in \mathcal{L}^n_{\mathrm{BL}}$. The induction to more complex α is routine, and left to the reader.

L → R: Suppose $O(B/A) \in \mathsf{w}^n$. To show that $\mathrm{Max}_{\preceq}(|A|_M) \subseteq |B|_M$, consider any $\langle x, X\rangle \in \mathrm{Max}_{\preceq}(|A|_M)$. By Lemma 35, $\langle x, X\rangle \in \mathrm{Max}_{\preceq}(|[A]|)$. By Lemma 34(ii) then, x is normal for A. Hence, $x \in [B]$, and then $\langle x, X\rangle \in |[B]|$. So $\langle x, X\rangle \in |B|_M$, Lemma 35. That suffices for $\mathrm{Max}_{\preceq}(|A|_M) \subseteq |B|_M$, and so for $M \models_{\mathrm{P}} O(B/A)$.

R → L: Suppose $M \models_{\mathrm{P}} O(B/A)$, so that $\mathrm{Max}_{\preceq}(|A|_M) \subseteq |B|_M$. By Lemma 35, $\mathrm{Max}_{\preceq}(|[A]|) \subseteq |[B]|$. To find $O(B/A) \in \mathsf{w}^n$, suppose, for *reductio*, $O(B/A) \notin \mathsf{w}^n$. Then $\Delta_A \cup \{\neg B\}$ is consistent, Lemma 4(ii), and so there is an $x \in W^{\mathrm{BL}^n}$ such that

$\Delta_A \cup \{\neg B\} \subseteq x$. Thus, x is normal for A. By Lemma 34(i), $\langle x, [A]\rangle \in \mathrm{Max}_{\leq}(|[A]|)$. So $\langle x, [A]\rangle \in |[B]|$, and thus $x \in [B]$, i.e., $B \in x$. But also $\neg B \in x$, so that $B \notin x$, a contradiction. Therefore, $O(B/A) \in \mathsf{w}^n$. ■

Lemma 37 *For finite $\mathcal{L}^n_{\mathrm{BL}}$, M is (i) finite and (ii) replete for $\mathcal{L}^n_{\mathrm{BL}}$.*

Proof. For (i), since $\mathcal{L}^n_{\mathrm{BL}}$ is finite, W^{BL^n} is finite. Since, for any $A \in \mathcal{L}^n_{\mathrm{BL}}$, $[A] \subseteq W^{\mathrm{BL}^n}$, the set of all such sets $[A]$ is also finite. Hence there are finitely many pairs $\langle x, X\rangle$ with $x \in W^{\mathrm{BL}^n}$ and $X = [A]$ for some $A \in \mathcal{L}^n_{\mathrm{BL}}$. Thus W is finite. For (ii), if, for $A \in \mathcal{L}^n_{\mathrm{BL}}$, $|A| \neq \varnothing$, then A is BL-consistent. By (RP), $\vdash P(\top/A)$ in DDL^n-c, so $P(\top/A) \in \mathsf{w}^n$. By Lemma 4(i), Δ_A is consistent, and so there is an $x \in W^{\mathrm{BL}^n}$ such that $\Delta_A \subseteq x$. Since x is thus normal for A, $\langle x, [A]\rangle \in W$. Since $A \in x$, $\langle x, [A]\rangle \in |[A]|$, and so, by Lemma 35, $\langle x, [A]\rangle \in |A|_M$. Hence $|A|_M \neq \varnothing$, as required for repletion. ■

Lemma 38 *$\leq$ is stoppered for $\mathcal{L}^n_{\mathrm{BL}}$.*

Proof. That W is finite and $\leq$ is transitive suffices for $\leq$ to be stoppered. Nevertheless, to enable this result to be readily applicable to the full infinite DDL-c, we also present a demonstration based on the structure of this model M.

Consider any $B \in \mathcal{L}^n_{\mathrm{BL}}$, and suppose $\langle x, X\rangle \in |B|_M$. Since $\langle x, X\rangle \in W$, there is an $A \in \mathcal{L}^n_{\mathrm{BL}}$ such that $X = [A]$ and x is normal for A. Since $\langle x, X\rangle \in |B|_M$, $\langle x, X\rangle \in |[B]|$, by Lemma 35, and so $x \in [B]$ and thus $B \in x$. Either $O(A/A \vee B) \in \mathsf{w}^n$ or $O(A/A \vee B) \notin \mathsf{w}^n$. For the first case, we show (a) $\langle x, X\rangle \in \mathrm{Max}_{\leq}(|B|_M)$; for the second case, we show (b) there is a $\langle y, Y\rangle \in \mathrm{Max}_{\leq}(|B|_M)$ such that $\langle y, Y\rangle \prec \langle x, X\rangle$. (a) and (b) together mean $\leq$ is stoppered for $\mathcal{L}^n_{\mathrm{BL}}$.

For (a), suppose $O(A/A \vee B) \in \mathsf{w}^n$. Since $\langle x, X\rangle \in |B|_M$, suppose, for *reductio*, there is some $\langle y, Y\rangle$ such that $\langle y, Y\rangle \in |B|_M$ such that $\langle y, Y\rangle \prec \langle x, X\rangle$, so that also $\langle y, Y\rangle \leq \langle x, X\rangle$. By Lemma 35, $\langle y, Y\rangle \in |[B]|$; so $y \in [B]$. Further, since $\langle y, Y\rangle \in W$, there is some C such that $Y = [C]$ and y is normal for C. Obviously $\langle x, X\rangle \neq \langle y, Y\rangle$. Since $\langle y, Y\rangle \leq \langle x, X\rangle$, for all D and E such that $Y = [D]$ and $X = [E]$, $O(D/D \vee E) \in \mathsf{w}^n$ and $y \notin [E]$. Thus, $O(C/A \vee C) \in \mathsf{w}^n$ and $y \notin [A]$. We now show $O(B \to A/C) \in \mathsf{w}^n$. Given $O(A/A \vee B) \in \mathsf{w}^n$, then $O(A \vee C/A \vee B) \in \mathsf{w}^n$, by (RW); likewise $O(A \vee C/A \vee C) \in \mathsf{w}^n$ by (Reflex) or (RW). Hence (i) $O(A \vee C/A \vee B \vee C) \in \mathsf{w}^n$ by (OR) and (LLE). Further, since $O(A/A \vee B) \in \mathsf{w}^n$, $O(A/(A \vee B) \wedge (A \vee B \vee C)) \in \mathsf{w}^n$ by (LLE). So $O((A \vee B) \to A/A \vee B \vee C) \in \mathsf{w}^n$ by (S), Lemma 5(1), whence (ii) $O(B \to A/A \vee B \vee C) \in \mathsf{w}^n$ by (RW). (i) and (ii) yield (iii) $O(B \to A/(A \vee C) \wedge (A \vee B \vee C)) \in \mathsf{w}^n$ by (CautMono), whence $O(B \to A/A \vee C) \in \mathsf{w}^n$ by (LLE). That with $O(C/A \vee C) \in \mathsf{w}^n$ yield $O(B \to A/C \wedge (A \vee C)) \in \mathsf{w}^n$ by (CautMono) again. From that $O(B \to A/C) \in \mathsf{w}^n$ by (LLE), as desired. Since y is normal for C, $B \to A \in y$, and since $B \in y$, then $A \in y$, so that $y \in [A]$, a contradiction. Therefore, there is no such $\langle y, Y\rangle$, and so $\langle x, X\rangle \in \mathrm{Max}_{\leq}(|B|_M)$.

For (b), suppose $O(A/A \vee B) \notin \mathsf{w}^n$. Then $\Delta_{A \vee B} \cup \{\neg A\}$ is consistent, Lemma 4(ii), and so there is a $y \in W^{\mathrm{BL}^n}$ such that $\Delta_{A \vee B} \cup \{\neg A\} \subseteq y$. Thus y is normal for $A \vee B$; so $\langle y, [A \vee B]\rangle \in W$. Since $A \vee B \in y$ and $\neg A \in y$, $B \in y$, so that $y \in [B]$. Thus, $\langle y, [A \vee B]\rangle \in |[B]|$. By Lemma 34(i), $\langle y, [A \vee B]\rangle \in \mathrm{Max}_{\leq}(|[A \vee B]|)$. It follows that $\langle y, [A \vee B]\rangle \in \mathrm{Max}_{\leq}(|[B]|)$, for if there were some $\langle z, Z\rangle \in |[B]|$ such that $\langle z, Z\rangle \prec \langle y, [A \vee B]\rangle$, then since $\langle z, Z\rangle \in |[A \vee B]|$, $\langle y, [A \vee B]\rangle \notin \mathrm{Max}_{\leq}(|[A \vee B]|)$, a contradiction. Since thus $\langle y, [A \vee B]\rangle \in \mathrm{Max}_{\leq}(|[B]|)$, then $\langle y, [A \vee B]\rangle \in \mathrm{Max}_{\leq}(|B|_M)$, by Lemma 35.

We show next that $\langle y, [A \vee B]\rangle \prec \langle x, X\rangle$. For that, consider any D and E such that $[A \vee B] = [D]$ and $X = [E] = [A]$. Since $O(A \vee B/A \vee B \vee A) \in \mathsf{w}^n$, by (Reflex) and (LLE), $O(D/D \vee E) \in \mathsf{w}^n$ by Lemma 31(iii). Further, since $\neg A \in y$, $y \notin [A]$, so $y \notin [E]$. That suffices for $\langle y, [A \vee B]\rangle \preceq \langle x, X\rangle$. Since $x \in [A]$ and $y \notin [A]$, $x \neq y$, so that $\langle y, [A \vee B]\rangle \neq \langle x, X\rangle$. Hence, $\langle x, X\rangle \npreceq \langle y, [A \vee B]\rangle$, since, if $\langle x, X\rangle \preceq \langle y, [A \vee B]\rangle$, then $x \notin [A \vee B]$ by clause (ii), and since $x \in [A]$, $x \in [A \vee B]$. Consequently, $\langle y, [A \vee B]\rangle \prec \langle x, X\rangle$, as required for this case. ■

Completeness now follows:

Theorem 39 *For all finite $n \geq 1$, (i) DDL^n-c is weakly complete with respect to the class of all P-models defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ that are finite and replete for $\mathcal{L}^n_{\mathrm{BL}}$ and whose relation $\preceq$ is reflexive and stoppered for $\mathcal{L}^n_{\mathrm{BL}}$. I.e., for any $\alpha \in \mathcal{L}^n_{\mathrm{DL}^a}$, if $\Vdash_{\mathrm{P}} \alpha$ for this class, then $\vdash \alpha$ in DDL^n-c. (ii) DDL^n-c is weakly complete with respect to the class of all P-models defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ that are finite and replete for $\mathcal{L}^n_{\mathrm{BL}}$ and whose relation $\preceq$ is reflexive and stoppered for $\mathcal{L}^n_{\mathrm{BL}}$, and also transitive.*

Proof. By much the same argument as for Theorem 28. Given $\nvdash \alpha$ in DDL^n-c, Lemma 36 entails there is a model $M = \langle W, \preceq, v\rangle$ such that $M \nVdash_{\mathrm{P}} \alpha$, while Lemma 37 ensures that M is finite and replete for $\mathcal{L}^n_{\mathrm{BL}}$, Lemma 33 ensures that $\preceq$ is reflexive and, for (ii), transitive, and Lemma 38 ensures M is stoppered for $\mathcal{L}^n_{\mathrm{BL}}$. ■

Part (ii), with transitivity, will be useful in the proof of Theorem 57 below.

As with DDL-a and DDL-b, much the same demonstration would establish the strong completeness of the full system DDL-c over the full infinite language for the class of infinite stoppered P-models. Finally, as with DDL^n-a and DDL^n-b,

Corollary 40 *For each finite $n \geq 1$, DDL^n-c has the finite model property in terms of P-models defined for $\mathcal{L}^n_{\mathrm{DL}^a}$.*

Corollary 41 *For each finite $n \geq 1$, DDL^n-c is decidable.*

That completes this stage for DDL^n-c.

5.1.3 Finite DDL^n-d

Although we have already established the equivalence of DDL-d with DSDL3 in §4, for later reference, especially in §6.2, it will be convenient now to demonstrate the soundness and completeness of the finite counterparts of DDL-d in terms of finite models. Here we need only sketch the argument since, when filled in, it will merely recapitulate that of §4. Nothing there required the infinitude of $\mathcal{L}_{\mathrm{BL}}$ and $\mathcal{L}_{\mathrm{DL}^a}$. We do, however, now give these results in terms of P-models, which are easier to describe for finite constructions.

Soundness for DDL^n-d was given in Theorem 15. For completeness, suppose w^n to be a maximal DDL^n-d consistent set of formulas from $\mathcal{L}^n_{\mathrm{DL}^a}$. As earlier, let W^{BL^n} be the set of maximal consistent sets of $\mathcal{L}^n_{\mathrm{BL}}$ formulas. As before too, $\Delta_A = \{B : O(B/A) \in \mathsf{w}^n\}$, and, for $x \in W^{\mathrm{BL}^n}$, x is normal for A iff $\Delta_A \subseteq x$. Let $M = \langle W, \preceq, v\rangle$ be such that:

- $W = W^{\mathrm{BL}^n}$,
- for $w, w' \in W$, $w \leq w'$ iff, for all B such that w' is normal for B, there is an A such that w is normal for A and $P(A/A \vee B) \in \mathsf{w}^n$,
- $v(p) = \{w \in W : p \in w\}$, for all atoms $p \in \mathcal{L}^n_{\mathrm{BL}}$.

With its definition of $\leq$, M plainly mimics R from §4.

Lemma 42 *M is a P-model (i) defined for $\mathcal{L}^n_{\mathrm{DL}^a}$, (ii) finite, and (iii) replete for $\mathcal{L}^n_{\mathrm{BL}}$, and (iv) $\leq$ is reflexive, transitive and total, as well as limited for $\mathcal{L}^n_{\mathrm{BL}}$. I.e., M is a P-model apt for DDL^n-d.*

Proof. (i)–(iii) are obvious. (iv) is by the arguments for Lemmas 6 and 9, *mutatis mutandis*. ■

By the argument for Lemma 7, w is normal for A iff $w \in \mathrm{Max}_{\leq}(|A|_M)$, and so, by the argument for Lemma 8,

Lemma 43 *For all $\alpha \in \mathcal{L}^n_{\mathrm{DL}^a}$, $\alpha \in \mathsf{w}^n$ iff $M \models_{\mathrm{P}} \alpha$.*

Hence,

Theorem 44 *For all finite $n \geq 1$, DDL^n-d is sound and complete with respect to the class of all P-models defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ that are finite and replete for $\mathcal{L}^n_{\mathrm{BL}}$ with relations $\leq$ that are reflexive, transitive and total, as well as limited for $\mathcal{L}^n_{\mathrm{BL}}$.*

Proof. By the standard arguments. ■

As with the other finite systems,

Corollary 45 *For each finite $n \geq 1$, DDL^n-d has the finite model property in terms of P-models defined for $\mathcal{L}^n_{\mathrm{DL}^a}$.*

Corollary 46 *For each finite $n \geq 1$, DDL^n-d is decidable.*

This completes Stage 1 of our journey.

5.2 Stage 2

With the completion of Stage 1, we have found that, for $\alpha \in \mathcal{L}^n_{\mathrm{DL}^a}$, if $\nvdash \alpha$ for DDL^n-a, -b, -c, or -d, then there is a P-model $M = \langle W, \leq, v\rangle$ defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ of the appropriate kind such that $M \not\models_{\mathrm{P}} \alpha$. In this second stage, we develop a method whereby to match the, possibly duplicative, worlds of W in M with corresponding valuations $\varphi \in V$. That done, we can derive a Hanssonian model/relation, R, of the appropriate kind such that likewise $\mathsf{R} \not\models_{\mathrm{H}} \alpha$. From there the equivalence of the full DDL systems with the DSDL logics follows. We apply these procedures first to demonstrate that DDL-a = DSDL1, DDL-c = DSDL2.5 and DDL-d = DSDL3. DDL-b and DSDL2 are more difficult, and so deserve a subsection of their own.

5.2.1 Full DDL-a, DDL-c, and DDL-d

Given finite $n \geq 1$, consider any P-model $M = \langle W, \leq, v\rangle$ defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ over $\mathcal{L}^n_{\mathrm{BL}}$ that is (i) finite, (ii) replete for $\mathcal{L}^n_{\mathrm{BL}}$, and (iii) with $\leq$ reflexive, and perhaps (iv) transitive and so stoppered for $\mathcal{L}^n_{\mathrm{BL}}$, and also perhaps (v) total. M may be redundant. In short, M is to be a model of the sort presented in Stage 1 above for DDL^n-a, DDL^n-c or DDL^n-d. Suppose W to contain exactly k many members, including all duplicates. Suppose them to be ordered as $\langle w_1, \ldots, w_k\rangle$ by some enumeration. Consider a set $R = \{r_1, \ldots, r_k\}$ of exactly k many atoms $r \in \mathcal{L}_{\mathrm{BL}}$ such that each $r \notin \mathcal{L}^n_{\mathrm{BL}}$. Suppose R is ordered as $\langle r_1, \ldots, r_k\rangle$ by some enumeration. These new atoms will be used as markers for the worlds $w_i \in W$; that is, each $r_i \in R$ marks, or is the marker of, $w_i \in W$ in their respective orderings. V is the set of all classical valuations defined over $\mathcal{L}_{\mathrm{BL}}$. Given that marking, we pick out certain select members, φ_{w_i}, of V to stand in place of the worlds $w_i \in W$. For $w_i \in W$ and $r_i \in R$ its marker, let φ_{w_i} be that $\varphi \in V$ such that

- $\varphi(p) = 1$ iff $w_i \in v(p)$, for all atoms $p \in \mathcal{L}^n_{\mathrm{BL}}$,
- $\varphi(r_i) = 1$, and
- $\varphi(s) = 0$ for all other atoms $s \in \mathcal{L}_{\mathrm{BL}}$, i.e., $s \notin \mathcal{L}^n_{\mathrm{BL}}$ and $s \neq r_i$.

Clearly there is such a φ_{w_i}, and only one, for each $w_i \in W$. We will say φ_{w_i} is 'marked for' w_i, by virtue of its verifying the atom r_i, the marker for w_i. φ_{w_i} corresponds to, or is a counterpart of, w_i, in the sense that the two agree on all formulas in $\mathcal{L}^n_{\mathrm{BL}}$.

Lemma 47 *For all $A \in \mathcal{L}^n_{\mathrm{BL}}$, $w_i \in |A|_M$ iff $\varphi_{w_i} \in |A|$.*

Proof. By an easy induction on $A \in \mathcal{L}^n_{\mathrm{BL}}$, left to the reader. ■

While each $w_i \in W$ has its counterpart $\varphi_{w_i} \in V$, there will, of course, be myriad other members of V that correspond to no such $w_i \in W$, not because they do not agree with some w_i on all $A \in \mathcal{L}^n_{\mathrm{BL}}$, but because they are not marked for that w_i in the requisite way. E.g., there will be $\varphi \in V$ such that $\varphi(r_1) = \varphi(r_2) = 1$, for $r_1, r_2 \in R$, or $\varphi(s) = 1$ for some $s \notin \mathcal{L}^n_{\mathrm{BL}}$ and $s \neq r$ for any $r \in R$. Let V^μ be the set of those valuations in V that are marked counterparts for worlds in W.

- $V^\mu = \{\varphi \in V : \text{there is a } w_i \in W \text{ such that } \varphi = \varphi_{w_i}\}$.

Lemma 48 *V^μ is finite.*

Proof. Obvious, given that W of M is finite. ■

Valuations $\varphi_{w_i} \in V^\mu$ that are marked for their counterparts w_i will play the role of those counterparts in a new irredundant H-model. Given $M = \langle W, \leq, v\rangle$ as described, let $\mathsf{R}_M \subseteq V \times V$ be defined from M, thus: For all $\varphi, \varphi' \in V$,

- $\varphi \mathsf{R}_M \varphi'$ iff either
  - (i) $\varphi \in V^\mu$ and $\varphi' \in V^\mu$, and $w_i \leq w_j$ if $\varphi = \varphi_{w_i}$ and $\varphi' = \varphi_{w_j}$, or
  - (ii) $\varphi' \notin V^\mu$.

Hence, all members of V^μ are ranked among themselves as their counterparts are ranked in M, while all members of V^μ are ranked strictly higher than non-members of V^μ. Those non-members are ranked equally with each other.

Lemma 49 *(i)* R_M *is reflexive, and (ii) if* $\leq$ *is transitive, then* R_M *is transitive. (iii) If* $\leq$ *is total, then* R_M *is total.*

Proof. (i) Reflexivity is obvious, given that $\leq$ is reflexive. (ii) For transitivity, if $\leq$ is transitive, suppose $\varphi \mathsf{R}_M \varphi'$ and $\varphi' \mathsf{R}_M \varphi''$. If $\varphi'' \notin V^\mu$, then immediately $\varphi \mathsf{R}_M \varphi''$, by clause (ii) of the definition of R_M. If $\varphi' \notin V^\mu$, then also $\varphi'' \notin V^\mu$, and again $\varphi \mathsf{R}_M \varphi''$. If $\varphi \notin V^\mu$, then $\varphi' \notin V^\mu$, by clause (ii), and so $\varphi'' \notin V^\mu$, and $\varphi \mathsf{R}_M \varphi''$. Suppose then all three are in V^μ. Then there is a $w_i \in W$ such that $\varphi = \varphi_{w_i}$ and a $w_j \in W$ such that $\varphi' = \varphi_{w_j}$ and a $w_l \in W$ such that $\varphi'' = \varphi_{w_l}$. Since $\varphi_{w_i} \mathsf{R}_M \varphi_{w_j}$ and $\varphi_{w_j} \mathsf{R}_M \varphi_{w_l}$, then by definition of R_M, clause (i), it follows that $w_i \leq w_j$ and $w_j \leq w_l$, whereupon $w_i \leq w_l$, since $\leq$ is transitive. That suffices for $\varphi_{w_i} \mathsf{R}_M \varphi_{w_l}$, or $\varphi \mathsf{R}_M \varphi''$, by clause (i).

(iii) For totality, suppose $\leq$ is total over W. If $\varphi, \varphi' \in V^\mu$ then either $\varphi \mathsf{R}_M \varphi'$ or $\varphi' \mathsf{R}_M \varphi$, by the totality of $\leq$. If $\varphi \notin V^\mu$ or $\varphi' \notin V^\mu$, then either $\varphi' \mathsf{R}_M \varphi$ or $\varphi \mathsf{R}_M \varphi'$ by clause (ii) of the definition of R_M. ∎

Lemma 50 *For all* $A \in \mathcal{L}^n_{\mathrm{BL}}$, $w_i \in \mathrm{Max}_\leq(|A|_M)$ *iff* $\varphi_{w_i} \in \mathrm{Max}_{\mathsf{R}_M}(|A|)$.

Proof. Suppose $A \in \mathcal{L}^n_{\mathrm{BL}}$. L → R: Suppose $w_i \in \mathrm{Max}_\leq(|A|_M)$. Since $w_i \in |A|_M$, $\varphi_{w_i} \in |A|$, by Lemma 47. We show $\varphi_{w_i} \in \mathrm{Max}_{\mathsf{R}_M}(|A|)$. Suppose not; suppose, for *reductio*, there is some $\varphi \in |A|$ such that $\varphi \mathsf{P}_M \varphi_{w_i}$. Since $\varphi_{w_i} \in V^\mu$, it must be that $\varphi \in V^\mu$. Hence there is a w_j such that $\varphi = \varphi_{w_j}$. By definition of R_M, $w_j \prec w_i$. Since $w_j \in |A|_M$, by Lemma 47, then $w_i \notin \mathrm{Max}_\leq(|A|_M)$, a contradiction. Hence there is no such $\varphi \in |A|$, and $\varphi_{w_i} \in \mathrm{Max}_{\mathsf{R}_M}(|A|)$.

R → L: Suppose $\varphi_{w_i} \in \mathrm{Max}_{\mathsf{R}_M}(|A|)$. Since $\varphi_{w_i} \in |A|$, $w_i \in |A|_M$, by Lemma 47. We show $w_i \in \mathrm{Max}_\leq(|A|_M)$. Suppose not; suppose, for *reductio*, there is some $w_j \in |A|_M$ such that $w_j \prec w_i$. Consider $\varphi_{w_j} \in V^\mu$. $\varphi_{w_j} \in |A|$, by Lemma 47. By definition of R_M, $\varphi_{w_j} \mathsf{P}_M \varphi_{w_i}$, but then $\varphi_{w_i} \notin \mathrm{Max}_{\mathsf{R}_M}(|A|)$, a contradiction. Hence, $w_i \in \mathrm{Max}_\leq(|A|_M)$. ∎

Lemma 51 *For all* $\alpha \in \mathcal{L}^n_{\mathrm{DL}^a}$, $M \models_{\mathrm{P}} \alpha$ *iff* $\mathsf{R}_M \models_{\mathrm{H}} \alpha$.

Proof. By induction on α. We show the basis, where $\alpha = O(B/A)$. The induction to more complex cases is routine and easy, and so left to the reader. Suppose $O(B/A) \in \mathcal{L}^n_{\mathrm{DL}^a}$, and thus $A, B \in \mathcal{L}^n_{\mathrm{BL}}$.

L → R: Suppose $M \models_{\mathrm{P}} O(B/A)$, so that $\mathrm{Max}_\leq(|A|_M) \subseteq |B|_M$. To show $\mathrm{Max}_{\mathsf{R}_M}(|A|) \subseteq |B|$, and thus $\mathsf{R}_M \models_{\mathrm{H}} O(B/A)$, consider any $\varphi \in \mathrm{Max}_{\mathsf{R}_M}(|A|)$. Since $\varphi \in |A|$, and M is replete for $\mathcal{L}^n_{\mathrm{BL}}$, there is a $w_i \in W$ such that $w_i \in |A|_M$. Consider $\varphi_{w_i} \in V^\mu$. $\varphi_{w_i} \in |A|$, by Lemma 47. For φ, either $\varphi \in V^\mu$ or $\varphi \notin V^\mu$. In the second case, $\varphi_{w_i} \mathsf{P}_M \varphi$, and then $\varphi \notin \mathrm{Max}_{\mathsf{R}_M}(|A|)$, a contradiction. Hence, $\varphi \in V^\mu$, and so there is a $w_j \in W$ such that $\varphi = \varphi_{w_j}$. Thus $\varphi_{w_j} \in \mathrm{Max}_{\mathsf{R}_M}(|A|)$. By Lemma 50, $w_j \in \mathrm{Max}_\leq(|A|_M)$. Hence, $w_j \in |B|_M$. From that, $\varphi_{w_j} \in |B|$, Lemma 47, i.e., $\varphi \in |B|$, which suffices for $\mathrm{Max}_{\mathsf{R}_M}(|A|) \subseteq |B|$, and so for $\mathsf{R}_M \models_{\mathrm{H}} O(B/A)$.

R → L: Suppose $\mathsf{R}_M \models_{\mathrm{H}} O(B/A)$, so that $\mathrm{Max}_{\mathsf{R}_M}(|A|) \subseteq |B|$. To show $M \models_{\mathrm{P}} O(B/A)$, i.e., that $\mathrm{Max}_\leq(|A|_M) \subseteq |B|_M$, consider any $w_i \in \mathrm{Max}_\leq(|A|_M)$. Consider $\varphi_{w_i} \in V^\mu$ that is marked for w_i. By Lemma 50, $\varphi_{w_i} \in \mathrm{Max}_{\mathsf{R}_M}(|A|)$. Then $\varphi_{w_i} \in |B|$, whence $w_i \in |B|_M$, by Lemma 47. That suffices for $\mathrm{Max}_{\preceq}(|A|_M) \subseteq |B|_M$, and so for $M \models_{\mathrm{P}} O(B/A)$. ■

That lemma establishes the equivalence of M and R_M with respect to formulas in $\mathcal{L}^n_{\mathrm{DL}^a}$. Naturally it is silent with respect to other formulas in $\mathcal{L}_{\mathrm{DL}^a}$ since M has nothing to say about them. This does, however, provide what we need to establish that DDL-a is equivalent to DSDL1.

Theorem 52 *DDL-a = DSDL1; i.e., for all $\alpha \in \mathcal{L}_{\mathrm{DL}^a}$ (i) if $\vdash \alpha$ in DDL-a, then $\Vdash_{\mathrm{H}} \alpha$ with respect to the class of all reflexive Hanssonian relations $\mathsf{R} \subseteq V \times V$, and conversely (ii) if $\Vdash_{\mathrm{H}} \alpha$ with respect to the class of all reflexive Hanssonian relations $\mathsf{R} \subseteq V \times V$, then $\vdash \alpha$ in DDL-a.*

Proof. (i) corresponds to the soundness of DDL-a with respect to Hanssonian H-models, see Theorem 1. (ii) is weak completeness for DDL-a. Suppose $\Vdash_{\mathrm{H}} \alpha$ for reflexive relations R, but that $\nvdash \alpha$ in DDL-a. Obviously there is a finite $n \geq 1$ such that $\lambda(\alpha) = n$. By Lemma 14, $\nvdash \alpha$ in DDL^n-a, and by Theorem 28(i), there is a reflexive P-model $M = \langle W, \preceq, v \rangle$ such that $M \not\models_{\mathrm{P}} \alpha$. Let R_M be defined from M as described. By Lemma 51, $\mathsf{R}_M \not\models_{\mathrm{H}} \alpha$. Moreover, R_M is reflexive, Lemma 49(i), and so a model apt for DSDL1. Hence, $\nVdash_{\mathrm{H}} \alpha$ for the class of DSDL1 relations, a contradiction. Consequently, if $\Vdash_{\mathrm{H}} \alpha$ for that class, then $\vdash \alpha$ in DDL-a. ■

At the end of §5.2.2, with Corollary 83, we will see that DDL-a is also sound and complete for all relations R that are transitive as well as reflexive.

For the equivalence of DDL-c and DSDL2.5, we need to show that R_M is stoppered. For that we must go beyond M's being stoppered since M would only be stoppered for $\mathcal{L}^n_{\mathrm{BL}}$ and R_M must be stoppered for the full infinite language $\mathcal{L}_{\mathrm{BL}}$. To achieve that, it would help if the field of R_M were finite, since finitude coupled with transitivity entails stoppering, and we know R_M is transitive if $\preceq$ is. But, of course, the field of R_M is not finite. To get around that, we consider first a relation that is defined for a finite field, namely R^μ_M, the restriction of R_M to V^μ. I.e.,

- For all $\varphi, \varphi' \in V$, $\varphi \mathsf{R}^\mu_M \varphi'$ iff $\varphi, \varphi' \in V^\mu$ and $\varphi \mathsf{R}_M \varphi'$.

Lemma 53 *If $\preceq$ of M is transitive, then R^μ_M is transitive.*

Proof. By the argument for Lemma 49(ii), in the case where $\varphi, \varphi', \varphi'' \in V^\mu$. ■

Lemma 54 *If $\preceq$ of M is transitive, then R^μ_M is stoppered.*

Proof. By Lemma 48, V^μ is finite. Since R^μ_M is transitive, if $\preceq$ is, Lemma 53, then R^μ_M is stoppered since transitivity and finitude of field suffice for stoppering. ■

Lemma 55 *For all $A \in \mathcal{L}_{\mathrm{BL}}$ and for all $\varphi \in V^\mu$, $\varphi \in \mathrm{Max}_{\mathsf{R}^\mu_M}(|A|)$ iff $\varphi \in \mathrm{Max}_{\mathsf{R}_M}(|A|)$.*

Proof. Suppose $\varphi \in V^\mu$, and for L → R: suppose $\varphi \in \mathrm{Max}_{\mathsf{R}^\mu_M}(|A|)$. Hence, $\varphi \in |A|$. Suppose, for *reductio*, $\varphi \notin \mathrm{Max}_{\mathsf{R}_M}(|A|)$. Then there is a $\varphi' \in |A|$ such that $\varphi' \mathsf{P}_M \varphi$. Since $\varphi' \mathsf{R}_M \varphi$, $\varphi' \in V^\mu$, by definition of R_M. Hence, $\varphi' \mathsf{R}^\mu_M \varphi$, and likewise, since not-($\varphi \mathsf{R}_M \varphi'$), not-($\varphi \mathsf{R}^\mu_M \varphi'$). Thus, $\varphi' \mathsf{P}^\mu_M \varphi$, in which case $\varphi \notin \mathrm{Max}_{\mathsf{R}^\mu_M}(|A|)$, a contradiction. Therefore, $\varphi \in \mathrm{Max}_{\mathsf{R}_M}(|A|)$. The argument R → L is much the same. ■

Lemma 56 *If $\leq$ of M is transitive, then R_M is stoppered.*

Proof. Suppose $\leq$ is transitive. Consider any $\varphi \in |A|$, for any $A \in \mathcal{L}_{\mathrm{BL}}$. $\varphi \in V^\mu$ or $\varphi \notin V^\mu$. In case $\varphi \in V^\mu$, then if $\varphi \notin \mathrm{Max}_{\mathsf{R}_M}(|A|)$, $\varphi \notin \mathrm{Max}_{\mathsf{R}^\mu_M}(|A|)$, by Lemma 55. Then, since R^μ_M is stoppered, Lemma 54, there is a $\varphi' \in \mathrm{Max}_{\mathsf{R}^\mu_M}(|A|)$ such that $\varphi' \mathsf{P}^\mu_M \varphi$. Since both $\varphi, \varphi' \in V^\mu$, $\varphi' \mathsf{P}_M \varphi$. Further, by Lemma 55, $\varphi' \in \mathrm{Max}_{\mathsf{R}_M}(|A|)$, which suffices for stoppering for this case. In case $\varphi \notin V^\mu$, then either $V^\mu \cap |A| = \varnothing$ or $V^\mu \cap |A| \neq \varnothing$. In the first case, then $\varphi \in \mathrm{Max}_{\mathsf{R}_M}(|A|)$ since there is no $\varphi' \in |A|$ such that $\varphi' \mathsf{P}_M \varphi$, and that suffices for stoppering for this case. In the second case, there is a $\varphi' \in V^\mu$ and $\varphi' \in |A|$. Since R^μ_M is stoppered, there is a $\varphi'' \in V^\mu$ such that $\varphi'' \in \mathrm{Max}_{\mathsf{R}^\mu_M}(|A|)$. For such a φ'', $\varphi'' \in \mathrm{Max}_{\mathsf{R}_M}(|A|)$ by Lemma 55 and also $\varphi'' \mathsf{P}_M \varphi$, by the definition of R_M, and that suffices for stoppering for this case too. ■

We can now conclude the equivalence of DDL-c and DSDL2.5, as well as of DDL-d and DSDL3 again.

Theorem 57 *DDL-c = DSDL2.5; i.e., for all $\alpha \in \mathcal{L}_{\mathrm{DL}^a}$ (i) if $\vdash \alpha$ in DDL-c, then $\Vdash_{\overline{\mathrm{H}}} \alpha$ with respect to the class of all Hanssonian relations $\mathsf{R} \subseteq V \times V$ that are reflexive and stoppered, and conversely (ii) if $\Vdash_{\overline{\mathrm{H}}} \alpha$ with respect to the class of all Hanssonian relations $\mathsf{R} \subseteq V \times V$ that are reflexive and stoppered, then $\vdash \alpha$ in DDL-c. (iii) Likewise for all relations R that are transitive as well as reflexive and stoppered.*

Proof. By the argument for Theorem 52 above, with Lemma 56 to assure that R_M is stoppered, given that M has a transitive relation $\leq$, as given by Theorem 39, part (ii), and also Lemma 49(ii) for transitivity of R_M for part (iii). ■

We include part (iii) to complement Theorem 82 for DDL-b and its Corollary 83 for DDL-a with regard to transitive relations R.

Theorem 58 *DDL-d = DSDL3; i.e., for all $\alpha \in \mathcal{L}_{\mathrm{DL}^a}$ (i) if $\vdash \alpha$ in DDL-d, then $\Vdash_{\overline{\mathrm{H}}} \alpha$ with respect to the class of all Hanssonian relations $\mathsf{R} \subseteq V \times V$ that are reflexive, transitive and total and also limited, and conversely (ii) if $\Vdash_{\overline{\mathrm{H}}} \alpha$ with respect to the class of all Hanssonian relations $\mathsf{R} \subseteq V \times V$ that are reflexive, transitive and total and also limited, then $\vdash \alpha$ in DDL-d.*

Proof. By the argument for Theorem 57, applying Lemma 49 to ensure R_M is both transitive and total when $\leq$ is, as given by Theorem 44. By the argument for Lemma 56, R_M is stoppered. Since being stoppered entails being limited, R_M is limited. ■

While this theorem essentially repeats Theorem 10 and its Corollary 12 in §4, I include it now for later reference, in §6.2. We may note that the relation R_M applied here is defined differently from the relation R applied for Theorem 10.

5.2.2 Full DDL-b

Before demonstrating that the full DDL-b is equivalent to DSDL2, it may be helpful to see why the method that yielded the equivalences for DDL-a, DDL-c and DDL-d with DSDL1, DSDL2.5 and DSDL3 in the previous subsection breaks down for this system, and thus why we must go to extra lengths to achieve our desired result.

Here is a simple example based on $n = 2$. Suppose a certain $M^2 = \langle W^2, \preceq^2, v^2 \rangle$, defined for $\mathcal{L}^2_{\mathrm{DL}^a}$, that is finite and replete for $\mathcal{L}^2_{\mathrm{BL}}$, but redundant, and whose $\preceq^2$ is reflexive and limited for $\mathcal{L}^2_{\mathrm{BL}}$. To be replete requires W^2 to have at least four members, but with redundancy there will be more. Suppose $W^2 = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}$, and suppose v^2 is such that w_5 is a duplicate of w_1, w_6 a duplicate of w_2, w_7 a duplicate of w_3 and w_8 a duplicate of w_4, and also, for convenience, suppose w_5, w_6, w_7, w_8 suffice for repletion, though that will not matter here. Repletion is not the issue. Suppose $\preceq^2$ is given entirely by:

$$w_1 \preceq^2 w_2 \quad w_2 \preceq^2 w_3 \quad w_3 \preceq^2 w_4 \quad w_4 \preceq^2 w_1$$
$$w \preceq^2 w, \text{ for all } w \in W^2$$

Thus, w_1, w_2, w_3, w_4 form a loop by $\prec^2$, and w_5, w_6, w_7, w_8 just stand by themselves. The latter suffices for M^2 to be limited for $\mathcal{L}^2_{\mathrm{BL}}$, though that will not really matter here either.

Now we try to apply the method developed for DDL-a, etc. to define an appropriate relation R_{M^2}. Consider the eight atoms $r_1, \ldots, r_8 \notin \mathcal{L}^2_{\mathrm{BL}}$, to mark each of $w_1, \ldots, w_8 \in W^2$. Define $\varphi_{w_1}, \ldots, \varphi_{w_8} \in V^\mu$ as before. With R_{M^2} also defined as before, it follows that among V^μ, besides reflexivity, just

$$\varphi_{w_1} \mathsf{R}_{M^2} \varphi_{w_2} \quad \varphi_{w_2} \mathsf{R}_{M^2} \varphi_{w_3} \quad \varphi_{w_3} \mathsf{R}_{M^2} \varphi_{w_4} \quad \varphi_{w_4} \mathsf{R}_{M^2} \varphi_{w_1}$$

and so there is a loop by P_{M^2} for φ_{w_1}, φ_{w_2}, φ_{w_3}, φ_{w_4}, while φ_{w_5}, φ_{w_6}, φ_{w_7}, φ_{w_8} stand alone. By the latter, the limit condition will hold for all $A \in \mathcal{L}^2_{\mathrm{BL}}$, given that M^2 is limited for $\mathcal{L}^2_{\mathrm{BL}}$. I.e., for all $A \in \mathcal{L}^2_{\mathrm{BL}}$, if $|A| \neq \varnothing$, then $\mathrm{Max}_{\mathsf{R}_{M^2}}(|A|) \neq \varnothing$. For DDL-b to be equivalent to DSDL2, however, we need R_{M^2} to be limited for all $A \in \mathcal{L}_{\mathrm{BL}}$, and not merely $\mathcal{L}^2_{\mathrm{BL}}$.

Consider then the formula $B = r_1 \vee r_2 \vee r_3 \vee r_4$. Then $\varphi_{w_1} \in |B|$, $\varphi_{w_2} \in |B|$, $\varphi_{w_3} \in |B|$ and $\varphi_{w_4} \in |B|$, but none of those is maximal for $|B|$ by R_{M^2}, because of the loop. Each of $\varphi_{w_5} \notin |B|$, $\varphi_{w_6} \notin |B|$, $\varphi_{w_7} \notin |B|$ and $\varphi_{w_8} \notin |B|$, so of course none of those is maximal for $|B|$. There will be countless other $\varphi \in |B|$, but they will not be maximal there either since they are not in V^μ and so $\varphi_{w_1} \mathsf{P}_{M^2} \varphi$, for any such φ. In short there is no $\varphi \in V$ such that $\varphi \in \mathrm{Max}_{\mathsf{R}_{M^2}}(|B|)$, and the limit condition fails. As a result, $\mathsf{R}_{M^2} \not\models_{\mathrm{H}} P(\top/B)$ even though $P(\top/B)$ is valid for DSDL2.

To rescue the limit condition, for an appropriate relation R_M based on an arbitrary finite M, we must be assured that M contains no such terminal loops by $\prec$. While one might try to prove completeness for DDL^n-b for finite loop-free models, it is difficult to see how that would go. If the relation $\preceq$ of M were transitive, then that would exclude loops by $\prec$. On the other hand, for finite models, a transitive relation would suffice not only for the limit condition but also for stoppering, which would verify (CautMono), which is not in DDL-b, cf. Theorem 3. To avoid stoppering, we now take a different course. We will find DDL^n-b to be complete for P-models with transitive relations, and thus loop-free, but we will let go of finiteness for such models.

The demonstration takes two steps. In Step 1, we establish finite DDL^n-b to be complete with respect to P-models, M^τ, defined for finite $\mathcal{L}^n_{\mathrm{DL}^a}$, that are limited for $\mathcal{L}^n_{\mathrm{BL}}$ and also have a transitive relation ('tau' for transitive); those models themselves will be infinite and infinitely redundant. In Step 2, through the method of marking worlds, we produce an irredundant, Hanssonian relation $\mathsf{R} \subseteq V \times V$ that is limited for the full $\mathcal{L}_{\mathrm{BL}}$ and equivalent to M^τ for $\mathcal{L}^n_{\mathrm{DL}^a}$. That done, it will follow that DDL-b = DSDL2.

Step 1: Transitive P-models

Given a P-model $M = \langle W, \preceq, v\rangle$, defined for $\mathcal{L}^n_{\mathrm{DL}^a}$, that is finite and replete for $\mathcal{L}^n_{\mathrm{BL}}$ with $\preceq$ reflexive and limited for $\mathcal{L}^n_{\mathrm{BL}}$, we now construct another equivalent P-model $M^\tau = \langle W^\tau, \preceq^\tau, v^\tau\rangle$, defined for $\mathcal{L}^n_{\mathrm{DL}^a}$, that is replete for $\mathcal{L}^n_{\mathrm{BL}}$, and whose relation $\preceq^\tau$ is not only reflexive and limited for $\mathcal{L}^n_{\mathrm{BL}}$ but also transitive. Unlike M, M^τ is infinite, though its infinitude is only denumerable.[8]

Given $M = \langle W, \preceq, v\rangle$ as described, I will now, contrary to prior practice, use a, b, c, etc., as variables for members of W and i, j, k as numerical variables. ω is, as usual, the least limit ordinal, which may be identified with the set of natural numbers. From M, define $M^\tau = \langle W^\tau, \preceq^\tau, v^\tau\rangle$:

- $W^\tau = \{\langle a, b, i\rangle : a, b \in W \text{ and } i \in \omega\rangle$.
- For $\langle a, b, i\rangle, \langle c, d, j\rangle \in W^\tau$, $\langle a, b, i\rangle \preceq^\tau \langle c, d, j\rangle$ iff, either

  [1] $\langle a, b, i\rangle = \langle c, d, j\rangle$,

  or

  [2] both (a) $b = d$ and $i \geq j$, and (b) either

  (b.1) $c \neq d$ and $a = c$,

  or

  (b.2) $c = d$ and $a \prec c$.
- $v^\tau(p) = \{\langle a, b, i\rangle \in W^\tau : a \in v(p)\}$, for all atoms $p \in \mathcal{L}^n_{\mathrm{BL}}$.

Henceforth, I will use x, y, z, etc. as variables for members of W^τ.

Lemma 59 *M^τ is a P-model defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ over $\mathcal{L}^n_{\mathrm{BL}}$.*

Proof. This follows immediately from M being such a model, i.e., $W^\tau \neq \varnothing$ because $W \neq \varnothing$, and v^τ is clearly defined for all, and only, atoms $p \in \mathcal{L}^n_{\mathrm{BL}}$ because v is so defined. ■

Lemma 60 *W^τ is denumerable.*

Proof. Since M is finite, i.e., W is finite, there are only finitely many pairs $\langle a, b\rangle$, for $a, b \in W$. For each such pair there are denumerably many points $x = \langle a, b, i\rangle$ in W^τ, but the union of finitely many denumerable sets is still denumerable. Hence, W^τ is denumerable. ■

[8] The procedures of this step work as well for the full DDL-b over an infinite language and also for DDL^n-a and DDL-a, to show they too are complete with respect to appropriate P-models with transitive relations. These procedures do not require M to be finite, or $\preceq$ to be limited. Nevertheless, we keep our focus now just on DDL^n-b, for finite $n \geq 1$. The following is a modification and generalization of a method X. Parent used to establish the completeness of Åqvist's systems **E** and **F** with respect to P-models with transitive betterness relations and applying Rule P. (Personal communication, though those results should appear in his [13]. Parent cites Schlechta [15], esp. Prop 2.83, as a source. I have not seen that work.)

Lemma 61 $\preceq^\tau$ *is (i) reflexive and (ii) transitive.*

Proof. (i) is trivial. For (ii), suppose $\langle a,b,i\rangle \preceq^\tau \langle c,d,j\rangle$ and $\langle c,d,j\rangle \preceq^\tau \langle e,f,k\rangle$. We show $\langle a,b,i\rangle \preceq^\tau \langle e,f,k\rangle$. If either of the supposed relations is by clause [1], then the result is immediate. Hence suppose both are by clause [2]. Then, by [2.a], $b = d = f$ and $i \geq j \geq k$. So $b = f$ and $i \geq k$, as required by clause [2.a]. From the first supposition, by [2.b], either (1) $c \neq d$ and $a = c$ or (2) $c = d$ and $a \prec c$. Consider (1). From the second supposition, either $e \neq f$ and $c = e$, in which case $a = e$, and since $e \neq f$, $\langle a,b,i\rangle \preceq^\tau \langle e,f,k\rangle$ by [2.b.1], or else $e = f$ and $c \prec e$, in which case, since $a = c$, $a \prec e$, and so $\langle a,b,i\rangle \preceq^\tau \langle e,f,k\rangle$ by [2.b.2]. Under case (2), with $c = d$ and $a \prec c$, again either $e \neq f$ and $c = e$, or $e = f$ and $c \prec e$. In the first case, since $c = d$ and $d = f$ and $c = e$, then $e = f$, a contradiction. So this is not a possible case. In the other case, since $a \prec c$ and $c = d$ and $d = f$ and $e = f$, $a \prec e$, which suffices for $\langle a,b,i\rangle \preceq^\tau \langle e,f,k\rangle$ by clause [2.b.2]. Hence, in all possible cases, $\langle a,b,i\rangle \preceq^\tau \langle e,f,k\rangle$, as required. ■

To establish the equivalence of the models, these lemmas are useful.

Lemma 62 *For all* $A \in \mathcal{L}^n_{\mathrm{BL}}$ *and all* $\langle a,b,i\rangle \in W^\tau$, *(i)* $\langle a,b,i\rangle \in |A|_{M^\tau}$ *iff* $a \in |A|_M$, *and (ii) for all* $j \in \omega$, $\langle a,b,i\rangle \in |A|_{M^\tau}$ *iff* $\langle a,b,j\rangle \in |A|_{M^\tau}$.

Proof. (i) is by an easy induction on A, left to the reader. (ii) follows immediately from (i). ■

Lemma 63 *For all* $A \in \mathcal{L}^n_{\mathrm{BL}}$ *and all* $\langle a,b,i\rangle \in W^\tau$, *if* $\langle a,b,i\rangle \in \mathrm{Max}_{\preceq^\tau}(|A|_{M^\tau})$, *then* $a = b$.

Proof. Suppose that $\langle a,b,i\rangle \in \mathrm{Max}_{\preceq^\tau}(|A|_{M^\tau})$, but $a \neq b$. Since $\langle a,b,i\rangle \in |A|_{M^\tau}$, $\langle a,b,i+1\rangle \in |A|_{M^\tau}$, by Lemma 62(ii). Trivially, $b = b$ and $i+1 \geq i$, also $a = a$. Hence, clauses [2.a] and [2.b.1] of the definition of $\preceq^\tau$ are met, so $\langle a,b,i+1\rangle \preceq^\tau \langle a,b,i\rangle$. Since $\langle a,b,i\rangle \in \mathrm{Max}_{\preceq^\tau}(|A|_{M^\tau})$, it follows that $\langle a,b,i\rangle \preceq^\tau \langle a,b,i+1\rangle$. Hence, by clause [2.a], $i \geq i+1$, which is absurd, of course. Therefore, it must be that $a = b$. ■

Lemma 64 *For all* $A \in \mathcal{L}^n_{\mathrm{BL}}$ *and all* $a \in W$ *and all* $i \in \omega$, $a \in \mathrm{Max}_{\preceq}(|A|_M)$ *iff* $\langle a,a,i\rangle \in \mathrm{Max}_{\preceq^\tau}(|A|_{M^\tau})$.

Proof. L → R: Suppose $a \in \mathrm{Max}_{\preceq}(|A|_M)$. So $a \in |A|_M$, and thus $\langle a,a,i\rangle \in |A|_{M^\tau}$, by Lemma 62(i). We show $\langle a,a,i\rangle \in \mathrm{Max}_{\preceq^\tau}(|A|_{M^\tau})$. Suppose not, i.e., suppose for *reductio* there is some $\langle b,c,j\rangle \in |A|_{M^\tau}$ such that $\langle b,c,j\rangle \prec^\tau \langle a,a,i\rangle$. Since $\langle b,c,j\rangle \in |A|_{M^\tau}$, $b \in |A|_M$. Obviously, $\langle b,c,j\rangle \neq \langle a,a,i\rangle$. Since $\langle b,c,j\rangle \preceq^\tau \langle a,a,i\rangle$, $c = a$ and $j \geq i$, by clause [2.a]. Since $a = a$, $b \prec a$ by clause [2.b.2]. But that is impossible since $a \in \mathrm{Max}_{\preceq}(|A|_M)$. Hence, there is no such $\langle b,c,j\rangle \in |A|_{M^\tau}$, and so $\langle a,a,i\rangle \in \mathrm{Max}_{\preceq^\tau}(|A|_{M^\tau})$.

R → L: Suppose $\langle a,a,i\rangle \in \mathrm{Max}_{\preceq^\tau}(|A|_{M^\tau})$. Since $\langle a,a,i\rangle \in |A|_{M^\tau}$, $a \in |A|_M$ by Lemma 62. We show $a \in \mathrm{Max}_{\preceq}(|A|_M)$. Suppose not, i.e., suppose for *reductio* there is some $b \in |A|_M$ such that $b \prec a$. Obviously $b \neq a$. Consider $\langle b,a,i\rangle$. $\langle b,a,i\rangle \in W^\tau$. Also, $\langle b,a,i\rangle \in |A|_{M^\tau}$. Trivially, $a = a$ and $i \geq i$, so clause [2.a] of the definition of $\preceq^\tau$ is met. Since $b \prec a$, $\langle b,a,i\rangle \preceq^\tau \langle a,a,i\rangle$ by clause [2.b.2]. Since $\langle a,a,i\rangle \in \mathrm{Max}_{\preceq^\tau}(|A|_{M^\tau})$, $\langle a,a,i\rangle \preceq^\tau \langle b,a,i\rangle$. Since $b \neq a$, then, by clause [2.b.1], $a = b$, a direct contradiction. Hence, there is no such $b \in |A|_M$, and so $a \in \mathrm{Max}_{\preceq}(|A|_M)$. ■

It is now easy to verify that M and M^τ are equivalent for $\mathcal{L}^n_{\mathrm{DL}^a}$.

Lemma 65 *For all formulas $\alpha \in \mathcal{L}^n_{\mathrm{DL}^a}$, $M \models_{\mathrm{P}} \alpha$ iff $M^\tau \models_{\mathrm{P}} \alpha$.*

Proof. By induction on α. We show only the case where $\alpha = O(B/A)$ since the others are easily done, and may be left to the reader.

L $\rightarrow$ R: Suppose $M \models_{\mathrm{P}} O(B/A)$, so that $\mathrm{Max}_{\leq}(|A|_M) \subseteq |B|_M$. To show $M^\tau \models_{\mathrm{P}} O(B/A)$, i.e., that $\mathrm{Max}_{\leq^\tau}(|A|_{M^\tau}) \subseteq |B|_{M^\tau}$, suppose some $\langle a, b, i\rangle \in \mathrm{Max}_{\leq^\tau}(|A|_{M^\tau})$. By Lemma 63, $a = b$. Hence, $\langle a, a, i\rangle \in \mathrm{Max}_{\leq^\tau}(|A|_{M^\tau})$. By Lemma 64, $a \in \mathrm{Max}_{\leq}(|A|_M)$. Hence, $a \in |B|_M$, and then, by Lemma 62, $\langle a, b, i\rangle \in |B|_{M^\tau}$, which suffices for $\mathrm{Max}_{\leq^\tau}(|A|_{M^\tau}) \subseteq |B|_{M^\tau}$ and so for $M^\tau \models_{\mathrm{P}} O(B/A)$.

R $\rightarrow$ L: Suppose $M^\tau \models_{\mathrm{P}} O(B/A)$, so that $\mathrm{Max}_{\leq^\tau}(|A|_{M^\tau}) \subseteq |B|_{M^\tau}$. Consider any $a \in \mathrm{Max}_{\leq}(|A|_M)$. By Lemma 64, $\langle a, a, i\rangle \in \mathrm{Max}_{\leq^\tau}(|A|_{M^\tau})$, for any $i \in \omega$. Thus, $\langle a, a, i\rangle \in |B|_{M^\tau}$, whereupon $a \in |B|_M$, by Lemma 62. That suffices for $\mathrm{Max}_{\leq}(|A|_M) \subseteq |B|_M$, and so for $M \models_{\mathrm{P}} O(B/A)$. ■

For DDL^n-b, it remains to show that M^τ is replete for $\mathcal{L}^n_{\mathrm{BL}}$ and limited for $\mathcal{L}^n_{\mathrm{BL}}$.

Lemma 66 *M^τ is (i) replete for $\mathcal{L}^n_{\mathrm{BL}}$ and (ii) limited for $\mathcal{L}^n_{\mathrm{BL}}$, if M is; i.e., for all $A \in \mathcal{L}^n_{\mathrm{BL}}$, if $|A|_{M^\tau} \neq \varnothing$ then $\mathrm{Max}_{\leq^\tau}(|A|_{M^\tau}) \neq \varnothing$.*

Proof. (i) follows directly from M being replete for $\mathcal{L}^n_{\mathrm{BL}}$ and Lemma 62. For (ii), suppose $A \in \mathcal{L}^n_{\mathrm{BL}}$ and $|A|_{M^\tau} \neq \varnothing$. Suppose $\langle a, b, i\rangle \in |A|_{M^\tau}$. By Lemma 62, $a \in |A|_M$; hence, $|A|_M \neq \varnothing$. Given that M is limited for $\mathcal{L}^n_{\mathrm{BL}}$, $\mathrm{Max}_{\leq}(|A|_M) \neq \varnothing$. Suppose $b \in \mathrm{Max}_{\leq}(|A|_M)$. Consider $\langle b, b, i\rangle \in W^\tau$, for any $i \in \omega$. By Lemma 64, $\langle b, b, i\rangle \in \mathrm{Max}_{\leq^\tau}(|A|_{M^\tau})$. Hence, $\mathrm{Max}_{\leq^\tau}(|A|_{M^\tau}) \neq \varnothing$, as required. ■

These results suffice for completeness.

Theorem 67 *For any finite $n \geq 1$, DDL^n-b is sound and weakly complete for all P-models, $M = \langle W, \leq, v\rangle$, defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ that are replete for $\mathcal{L}^n_{\mathrm{BL}}$ and whose relation, $\leq$, is reflexive, transitive, and limited for $\mathcal{L}^n_{\mathrm{BL}}$.*

Proof. Soundness has been done, Theorem 15. For weak completeness, we argue as usual. Suppose $\nvdash \alpha$ in DDL^n-b. By Theorem 28(ii), there is a model $M = \langle W, \leq, v\rangle$ that is finite and replete for $\mathcal{L}^n_{\mathrm{BL}}$, with $\leq$ both reflexive and limited for $\mathcal{L}^n_{\mathrm{BL}}$, and is such that $M \not\models_{\mathrm{P}} \alpha$. Construct M^τ from M as described. By Lemmas 59, 61 and 66, M^τ is defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ and is replete for $\mathcal{L}^n_{\mathrm{BL}}$ and its $\leq^\tau$ is reflexive and transitive and limited for $\mathcal{L}^n_{\mathrm{BL}}$, and by Lemma 65, $M^\tau \not\models_{\mathrm{P}} \alpha$. Hence, $\not\Vdash_{\mathrm{P}} \alpha$ for that class of P-models. Accordingly, if $\Vdash_{\mathrm{P}} \alpha$ for that class, then $\vdash \alpha$ in DDL^n-b. ■

That is what was primarily to be proved for this step of the argument regarding DDL-b. While the result has intrinsic interest, perhaps, its purpose is to be applied in the next step, where we will see that M^τ based on M is more nuanced than merely having a transitive relation and being limited for $\mathcal{L}^n_{\mathrm{BL}}$.

Step 2: Transitive H-models

We now seek a Hanssonian relation R that is equivalent to M^τ for $\mathcal{L}^n_{\mathrm{DL}^a}$ and is limited for the full, infinite language $\mathcal{L}_{\mathrm{BL}}$; in passing, it is also transitive. To this end, we adapt the method of marking worlds that was applied earlier for DDL-a, DDL-c and DDL-d.

Given $M = \langle W, \preceq, v\rangle$, defined for $\mathcal{L}^n_{\mathrm{DL}^a}$, that is finite and replete for $\mathcal{L}^n_{\mathrm{BL}}$ and whose relation $\preceq$ is reflexive and limited for $\mathcal{L}^n_{\mathrm{BL}}$, let $M^\tau = \langle W^\tau, \preceq^\tau, v^\tau\rangle$ be derived from M as described above in Step 1. As we have seen, M^τ is defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ and replete for $\mathcal{L}^n_{\mathrm{BL}}$ and its relation $\preceq^\tau$ is not only reflexive and limited for $\mathcal{L}^n_{\mathrm{BL}}$ but also transitive. Also, for all $\alpha \in \mathcal{L}^n_{\mathrm{DL}^a}$, $M^\tau \models_{\mathrm{P}} \alpha$ iff $M \models_{\mathrm{P}} \alpha$. Further W^τ is denumerable.

Because W^τ is only denumerable, there are enough atoms in the denumerably infinite language $\mathcal{L}_{\mathrm{BL}}$ to mark each of the points $x \in W^\tau$ in much the same manner as before. Let $R = \{r : r$ is an atom of $\mathcal{L}_{\mathrm{BL}}$ and $r \notin \mathcal{L}^n_{\mathrm{BL}}\}$. Since $\mathcal{L}^n_{\mathrm{BL}}$ is finite, there is such an R, and it is denumerable. Let R be ordered as $\langle r_1, \ldots, r_i, \ldots\rangle$ by some enumeration, and let W^τ likewise be ordered as $\langle x_1, \ldots, x_i, \ldots\rangle$ by an enumeration. This establishes a one-one mapping between R and W^τ. We will say that each r_i in its ordering marks, or is the marker for, x_i in its, where the subscript indicates the position of r_i or x_i in their enumerations. Thus the subscripts i, j, etc. on x_i, x_j, etc. are quite independent of the internal numerical indexes of these points, i.e., when $x_i = \langle a, b, k\rangle$, i has nothing to do with k, except arithmetic, of course. Clearly, every atom $r \in R$ marks some $x \in W^\tau$, and every $x \in W^\tau$ is marked by some $r \in R$, and these are unique. When $x = x_i$ is given in context, I may write 'r_x' for r_i, the marker for x.

As in §5.2.1, marking worlds this way enables a correspondence between certain valuations $\varphi \in V$ and points $x_i \in W^\tau$. For each $x_i \in W^\tau$, let φ_{x_i} be that $\varphi \in V$ such that

- $\varphi(p) = 1$ iff $x_i \in v^\tau(p)$, for all atoms $p \in \mathcal{L}^n_{\mathrm{BL}}$,
- $\varphi(r_i) = 1$, and
- $\varphi(r_j) = 0$ for all other atoms $r_j \in R$ such that $r_j \neq r_i$.

As before, there is clearly such a φ_{x_i}, and only one, for each $x_i \in W^\tau$. We say that φ_{x_i} is marked for x_i, or is the marked counterpart of x_i, by virtue of its verifying r_i, and only r_i, from R, while agreeing with x_i on all formulas $A \in \mathcal{L}^n_{\mathrm{BL}}$.

Lemma 68 *For all $r \in R$, if $\varphi_{x_i} \in |r|$ and $\varphi_{x_j} \in |r|$, then $x_i = x_j$.*

Proof. Suppose $\varphi_{x_i} \in |r|$ and $\varphi_{x_j} \in |r|$, but $x_i \neq x_j$. By the latter, $i \neq j$ in the enumeration of W^τ. Hence, $r_i \neq r_j$ in the ordering of R. Since $\varphi_{x_i} \in |r|$, $r = r_i$, and since $\varphi_{x_j} \in |r|$, $r = r_j$, and thus $r_i = r_j$, a contradiction. Hence, $x_i = x_j$. ■

Lemma 69 *For all $A \in \mathcal{L}^n_{\mathrm{BL}}$, $x_i \in |A|_{M^\tau}$ iff $\varphi_{x_i} \in |A|$.*

Proof. By an easy induction on $A \in \mathcal{L}^n_{\mathrm{BL}}$, given the specification of φ_{x_i} from x_i. ■

Likewise, as before, let

- $V^\mu = \{\varphi \in V :$ there is an $x \in W^\tau$ and $\varphi = \varphi_x\}$.

Lemma 70 *V^μ is denumerable.*

Proof. Immediate from Lemma 60. ■

We now define R_{M^τ} much as previously. For $\varphi, \varphi' \in V$, let

- $\varphi \mathsf{R}_{M^\tau} \varphi'$ iff either
  - (i) both $\varphi \in V^\mu$ and $\varphi' \in V^\mu$, and $x_i \leq^\tau x_j$ when $\varphi = \varphi_{x_i}$ and $\varphi' = \varphi_{x_j}$, or
  - (ii) $\varphi' \notin V^\mu$.

Lemma 71 R_{M^τ} *is reflexive and transitive.*

Proof. Reflexivity is immediate from $\leq^\tau$ being reflexive. Transitivity follows, *mutatis mutandis*, by the argument for Lemma 49, given that $\leq^\tau$ is transitive. ■

Lemma 72 *For all* $A \in \mathcal{L}^n_{\mathrm{BL}}$, $x_i \in \mathrm{Max}_{\leq^\tau}(|A|_{M^\tau})$ *iff* $\varphi_{x_i} \in \mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|)$.

Proof. By the argument for Lemma 50. ■

R_{M^τ} and M^τ are equivalent with regard to $\mathcal{L}^n_{\mathrm{DL}^a}$, in the sense that:

Lemma 73 *For all* $\alpha \in \mathcal{L}^n_{\mathrm{DL}^a}$, $M^\tau \models_{\mathrm{P}} \alpha$ *iff* $\mathsf{R}_{M^\tau} \models_{\mathrm{H}} \alpha$.

Proof. By the argument for Lemma 51. ■

We now need to establish that R_{M^τ} is limited for all of $\mathcal{L}_{\mathrm{BL}}$, that if $A \in \mathcal{L}_{\mathrm{BL}}$, then if $|A| \neq \varnothing$ then $\mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|) \neq \varnothing$. We distinguish cases depending on A. Case 1, $A \in \mathcal{L}^n_{\mathrm{BL}}$; Case 2, $A \notin \mathcal{L}^n_{\mathrm{BL}}$.

Lemma 74 (Case 1) *If* $A \in \mathcal{L}^n_{\mathrm{BL}}$ *then if* $|A| \neq \varnothing$ *then* $\mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|) \neq \varnothing$.

Proof. Given $A \in \mathcal{L}^n_{\mathrm{BL}}$ and $|A| \neq \varnothing$, then $|A|_{M^\tau} \neq \varnothing$, since M^τ is replete for $\mathcal{L}^n_{\mathrm{BL}}$. Since M^τ is limited for $\mathcal{L}^n_{\mathrm{BL}}$, $\mathrm{Max}_{\leq^\tau}(|A|_{M^\tau}) \neq \varnothing$. Suppose $x_i \in \mathrm{Max}_{\leq^\tau}(|A|_{M^\tau})$, then $\varphi_{x_i} \in \mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|)$, by Lemma 72. Hence, $\mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|) \neq \varnothing$. ■

We now consider Case 2 where $A \notin \mathcal{L}^n_{\mathrm{BL}}$, and so there are atoms $r \in R$ that are subformulas of A. Let

- $R(A) = \{r \in R : r \text{ is a subformula of } A\}$.

Obviously $R(A)$ is finite and, for this case, nonempty.

Given $|A| \neq \varnothing$, A is consistent. By classical logic, A is equivalent to a formula A' in disjunctive normal form (DNF), so that $A' = B_1 \vee \cdots \vee B_m$, where each disjunct, B_i, is a consistent conjunction of literals, i.e., atoms or their negations, from $\mathcal{L}_{\mathrm{BL}}$. Moreover, we now require of each such B_i, (i) for every atom $p \in \mathcal{L}^n_{\mathrm{BL}}$, either p or $\neg p$ is a conjunct of B_i, and not both, of course. This is possible since $\mathcal{L}^n_{\mathrm{BL}}$ is finite, containing exactly n many atoms. We also require (ii) for every atom $r \in R(A)$, either r or $\neg r$ is a conjunct of B_i, and not both. This too is possible since $R(A)$ is finite. Thus, each such B_i is equivalent to a formula $P_i \wedge R_i$, where P_i is a complete conjunction of literals from $\mathcal{L}^n_{\mathrm{BL}}$, one for every atom of $\mathcal{L}^n_{\mathrm{BL}}$, and R_i is a complete conjunction of literals from $R(A)$, one for every atom in $R(A)$. Let

- $\delta(A) = \{B_i : B_i \text{ is a disjunct of } A'\}$.

Like Gaul, $\delta(A)$ is divided into three parts. Let

- $\delta_0(A) = \{B_i \in \delta(A) : B_i$ is equivalent to $P_i \wedge R_i$ and all the conjuncts in R_i are negative, i.e., for all ℓ that are conjuncts of R_i, $\ell = \neg r$ for some $r \in R(A)\}$.
- $\delta_1(A) = \{B_i \in \delta(A) : B_i$ is equivalent to $P_i \wedge R_i$ and exactly one of the conjuncts of R_i is positive, i.e., there is one ℓ that is a conjunct of R_i such that $\ell = r$ for some $r \in R(A)$, and all the rest are negative$\}$.
- $\delta_2(A) = \{B_i \in \delta(A) : B_i$ is equivalent to $P_i \wedge R_i$ and more than one of the conjuncts of R_i is positive, i.e., there are ℓ and ℓ', both conjuncts of R_i, and $\ell = r$ for some $r \in R(A)$ and $\ell' = r'$ for some $r' \in R(A)$ and $r \neq r'\}$.

Clearly, each $\delta_i(A)$, for $i \in \{0, 1, 2\}$, is finite, since A' has finitely many disjuncts. Of course, any of these might be empty, though not all, given that A is consistent.

Given $\delta_i(A)$ for $i \in \{0, 1, 2\}$, let $A_i = \bigvee \delta_i(A)$, except in case $\delta_i(A) = \varnothing$, then let $A_i = p \wedge \neg p$, for some atom $p \in \mathcal{L}^n_{\mathrm{BL}}$. Thus, A is equivalent to $A_0 \vee A_1 \vee A_2$.

By the nature of the members of $\delta_2(A)$, we know this:

Lemma 75 *If $\varphi \in |A_2|$ then $\varphi \notin V^\mu$.*

Proof. Obvious. ■

We know less about those $\varphi \in |A_0|$ or $\varphi \in |A_1|$; these might, or might not, be in V^μ. We can, however, focus attention on those that are. For the given A, let

- $V^\mu_0(A) = V^\mu \cap |A_0|$;
- $V^\mu_1(A) = V^\mu \cap |A_1|$.

In light of Lemma 75, these exhaust $V^\mu \cap |A|$, and they are exclusive.

Lemma 76 *(i) $V^\mu \cap |A| = V^\mu_0(A) \cup V^\mu_1(A)$, and (ii) $V^\mu_0(A) \cap V^\mu_1(A) = \varnothing$.*

We note further that, given that $R(A)$ is finite,

Lemma 77 *$V^\mu_1(A)$ is finite.*

That is not so for $V^\mu_0(A)$.

With those in place, we now distinguish two further cases. In Case 2.a, $V^\mu \cap |A| = \varnothing$; in Case 2.b, $V^\mu \cap |A| \neq \varnothing$.

Lemma 78 (Case 2.a) *If $A \notin \mathcal{L}^n_{\mathrm{BL}}$ and $V^\mu \cap |A| = \varnothing$, then if $|A| \neq \varnothing$ then* $\mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|) \neq \varnothing$.

Proof. Suppose $A \notin \mathcal{L}^n_{\mathrm{BL}}$ and $V^\mu \cap |A| = \varnothing$ and $|A| \neq \varnothing$. Consider $\varphi \in |A|$. So $\varphi \notin V^\mu$. Suppose, for *reductio*, $\varphi \notin \mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|)$. Then there is a $\varphi' \in |A|$ such that $\varphi' \mathsf{P}_{M^\tau} \varphi$. By the definition of R_{M^τ}, that requires $\varphi' \in V^\mu$; so $\varphi' \in V^\mu \cap |A|$ and $V^\mu \cap |A| \neq \varnothing$, a contradiction. Hence, $\varphi \in \mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|)$ and $\mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|) \neq \varnothing$. ■

For Case 2.b, with $V^\mu \cap |A| \neq \varnothing$, we again distinguish two cases: Case 2.b.1, $V^\mu_0(A) \neq \varnothing$, and Case 2.b.2, $V^\mu_0(A) = \varnothing$.

Lemma 79 (Case 2.b.1) *If $A \notin \mathcal{L}^n_{\mathrm{BL}}$ and $V^\mu \cap |A| \neq \varnothing$, then if $V^\mu_0(A) \neq \varnothing$, then* $\mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|) \neq \varnothing$.

Proof. Suppose $A \notin \mathcal{L}^n_{\mathrm{BL}}$ and $V^\mu \cap |A| \neq \varnothing$, and suppose $V^\mu_0(A) \neq \varnothing$. Let $\varphi \in V^\mu_0(A)$. Thus $\varphi \in V^\mu$ and $\varphi \in |A_0|$, where $A_0 = P_1 \wedge R_1 \vee \cdots \vee P_k \wedge R_k$, in which each P_i is a complete conjunction of literals from $\mathcal{L}^n_{\mathrm{BL}}$ and R_i is a complete conjunction of negated atoms from $R(A)$. Thus, all the R_i's must be the same. Call that R_0. Hence, A_0 is equivalent to $(P_1 \vee \cdots \vee P_k) \wedge R_0$. Let $P_0 = P_1 \vee \cdots \vee P_k$. $P_0 \in \mathcal{L}^n_{\mathrm{BL}}$. Since $\varphi \in |A_0|$, $\varphi \in |P_0|$, so that $|P_0| \neq \varnothing$. Since M^τ is replete for $\mathcal{L}^n_{\mathrm{BL}}$, $|P_0|_{M^\tau} \neq \varnothing$, and since M^τ is limited for $\mathcal{L}^n_{\mathrm{BL}}$, $\mathrm{Max}_{\leq^\tau}(|P_0|_{M^\tau}) \neq \varnothing$. Suppose $x \in \mathrm{Max}_{\leq^\tau}(|P_0|_{M^\tau})$. By Lemma 63, we know $x = \langle a, a, i\rangle$, for some $a \in W$ and $i \in \omega$. By Lemma 64, $\langle a, a, j\rangle \in \mathrm{Max}_{\leq^\tau}(|P_0|_{M^\tau})$ for every $j \in \omega$.

Let $j^* \in \omega$ be the least number such that for every $r \in R(A)$, if $r = r_i$ in the enumeration of R, and r_i marks $x_i = \langle c, d, k\rangle$ in the enumeration of W^τ, then $j^* > k$. Since $R(A)$ is finite, and every $r \in R(A)$ marks a unique $x_i \in W^\tau$, there must be such a j^*. As noted, $\langle a, a, j^*\rangle \in \mathrm{Max}_{\leq^\tau}(|P_0|_{M^\tau})$. Let $y^* = \langle a, a, j^*\rangle$, and consider $\varphi_{y^*} \in V^\mu$. By Lemma 69, $\varphi_{y^*} \in |P_0|$. Further, $\varphi_{y^*} \in |R_0|$, for suppose not, i.e., suppose, for *reductio*, $\varphi_{y^*} \notin |R_0|$. Thus $\varphi_{y^*} \in |\neg R_0|$. Since R_0 is a conjunction $\neg r_a \wedge \ldots \wedge \neg r_b$, one conjunct for each atom in $R(A)$, $\neg R_0$ is equivalent to $r_a \vee \cdots \vee r_b$, one disjunct for each atom in $R(A)$. Hence, there is an $r \in R(A)$ such that $\varphi_{y^*} \in |r|$. Since $\varphi_{y^*} \in V^\mu$, there could be only one. Suppose that r marks point $z \in W^\tau$, where $z = \langle c, d, k\rangle$ for some $c, d \in W$ and $k \in \omega$. Thus $r = r_z$, so that $\varphi_{y^*} \in |r_z|$. Given z, consider φ_z marked for z by r_z. Obviously $\varphi_z \in |r_z|$. Hence, by Lemma 68, $y^* = z$, i.e., $\langle a, a, j^*\rangle = \langle c, d, k\rangle$. Thus $j^* = k$. On the other hand, given the specification of j^*, since $r_z \in R(A)$ and $z = \langle c, d, k\rangle$, $j^* > k$, a contradiction. Hence, $\varphi_{y^*} \in |R_0|$. Thus $\varphi_{y^*} \in |P_0 \wedge R_0|$, and so $\varphi_{y^*} \in A_0$, which entails $\varphi_{y^*} \in |A|$. We show it to be maximal in $|A|$ by R_{M^τ}.

Suppose, for *reductio*, $\varphi_{y^*} \notin \mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|)$, so that there is some $\varphi \in |A|$ such that $\varphi \mathsf{P}_{M^\tau} \varphi_{y^*}$. Since $\varphi_{y^*} \in V^\mu$, $\varphi \in V^\mu$, by the definition of R_{M^τ}. Hence, there is a $z \in W^\tau$ such that $\varphi = \varphi_z$, where $z = \langle c, d, k\rangle$ for some $c, d \in W$ and $k \in \omega$. Since $\varphi_{y^*}, \varphi_z \in V^\mu$ and $\varphi_z \mathsf{P}_{M^\tau} \varphi_{y^*}$, $z <^\tau y^*$. $\varphi_z \in V^\mu_1(A)$ or $\varphi_z \notin V^\mu_1(A)$. In the first case, with $\varphi_z \in V^\mu_1(A)$, and so $\varphi_z \in |A_1|$, there must be a $B_l \in \delta_1(A)$ such that $\varphi_z \in |B_l|$. B_l is equivalent to a conjunction $P_l \wedge R_l$, in which R_l is a conjunction of literals from the atoms of $R(A)$ and exactly one of those is positive. Since $\varphi_z \in |r_z|$ and $\varphi_z \in R_l$, it must be that r_z is that conjunct, and so $r_z \in R(A)$. Given $z <^\tau y^*$, $z \leq^\tau y^*$, i.e., $\langle c, d, k\rangle \leq^\tau \langle a, a, j^*\rangle$. Hence, $k \geq j^*$, by part [2.a] of the definition of $\leq^\tau$. By the specification of j^*, however, $j^* > k$, a contradiction. In the second case, with $\varphi_z \notin V^\mu_1(A)$, then $\varphi_z \in V^\mu_0(A)$, by Lemma 76. Then $\varphi_z \in |A_0|$ and so $\varphi_z \in |P_0|$, in which case $z \in |P_0|_{M^\tau}$, by Lemma 69. But then $y^* \notin \mathrm{Max}_{\leq^\tau}(|P_0|_{M^\tau})$, another contradiction. Hence, there is no such φ, and so $\varphi_{y^*} \in \mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|)$. Consequently, $\mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|) \neq \varnothing$. ■

Lemma 80 (Case 2.b.2) *If $A \notin \mathcal{L}^n_{\mathrm{BL}}$ and $V^\mu \cap |A| \neq \varnothing$, then if $V^\mu_0(A) = \varnothing$, then* $\mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|) \neq \varnothing$.

Proof. Suppose $A \notin \mathcal{L}^n_{\mathrm{BL}}$ and $V^\mu \cap |A| \neq \varnothing$, but $V^\mu_0(A) = \varnothing$. Then $V^\mu_1(A) \neq \varnothing$, by Lemma 76. By Lemma 77, $V^\mu_1(A)$ is finite. Since, by Lemma 71, R_{M^τ} is transitive, $\mathrm{Max}_{\mathsf{R}_{M^\tau}}(V^\mu_1(A)) \neq \varnothing$, for transitivity over a nonempty finite set guarantees a maximal member of the set. Suppose $\varphi^* \in \mathrm{Max}_{\mathsf{R}_{M^\tau}}(V^\mu_1(A))$. Since $\varphi^* \in V^\mu_1(A)$, $\varphi^* \in |A_1|$, and so $\varphi^* \in |A|$. We show it to be maximal in $|A|$. Suppose, for *reductio*, $\varphi^* \notin \mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|)$, so that there is some $\varphi \in |A|$ such that $\varphi \mathsf{P}_{M^\tau}\varphi^*$. Since $\varphi^* \in V^\mu$, $\varphi \in V^\mu$, by the definition of R_{M^τ}. Hence, $\varphi \in V^\mu \cap |A|$. Since, by the opening supposition, $\varphi \notin V_0^\mu(A)$, then $\varphi \in V_1^\mu(A)$, by Lemma 76. But then $\varphi^* \notin \mathrm{Max}_{\mathsf{R}_{M^\tau}}(V_1^\mu(A))$, a contradiction. Hence there is no such φ, and so $\varphi^* \in \mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|)$. Thus again, $\mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|) \neq \varnothing$. ■

These cases ensure that R_{M^τ} is limited.

Lemma 81 R_{M^τ} *is limited; i.e., for all $A \in \mathcal{L}_{\mathrm{BL}}$, if $|A| \neq \varnothing$, then $\mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|) \neq \varnothing$.*

Proof. Suppose $A \in \mathcal{L}_{\mathrm{BL}}$ and $|A| \neq \varnothing$. (i) If $A \in \mathcal{L}^n_{\mathrm{BL}}$, then $\mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|) \neq \varnothing$, by Case 1, Lemma 74. (ii) If $A \notin \mathcal{L}^n_{\mathrm{BL}}$, then (a) if $V^\mu \cap |A| = \varnothing$, then $\mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|) \neq \varnothing$, by Case 2.a, Lemma 78. On the other hand, (b) if $V^\mu \cap |A| \neq \varnothing$, then (1) if $V_0^\mu(A) \neq \varnothing$, then $\mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|) \neq \varnothing$, by Case 2.b.1, Lemma 79. But (2) if $V_0^\mu(A) = \varnothing$, then $\mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|) \neq \varnothing$, by Case 2.b.2, Lemma 80. Since those are all the possible cases, if $|A| \neq \varnothing$, then $\mathrm{Max}_{\mathsf{R}_{M^\tau}}(|A|) \neq \varnothing$, as required. ■

This completes what was needed in Step 2.

Theorem 82 *DDL-b = DSDL2; i.e., for all $\alpha \in \mathcal{L}_{\mathrm{DL}^a}$, (i) if $\vdash \alpha$ in DDL-b, then $\Vdash_{\mathrm{H}} \alpha$ with respect to the class of all Hanssonian relations $\mathsf{R} \subseteq V \times V$ that are reflexive and limited, and conversely (ii) if $\Vdash_{\mathrm{H}} \alpha$ with respect to the class of all Hanssonian relations $\mathsf{R} \subseteq V \times V$ that are reflexive and limited, then $\vdash \alpha$ in DDL-b. Likewise for all relations R that are transitive as well as reflexive and limited.*

Proof. (i) is the soundness of DDL-b, Theorem 1. (ii) is the weak completeness of DDL-b. We argue as previously, though with an extra step. Suppose $\Vdash_{\mathrm{H}} \alpha$ for the class of relations R that are reflexive and limited, but that $\nvdash \alpha$ in DDL-b. There must be some finite n such that $\lambda(\alpha) = n$. By Lemma 14, $\nvdash \alpha$ in DDL^n-b. By Theorem 28(ii), there is a P-model $M = \langle W, \preceq, v\rangle$, defined for $\mathcal{L}^n_{\mathrm{DL}^a}$, that is finite and replete for $\mathcal{L}^n_{\mathrm{BL}}$ and whose relation $\preceq$ is reflexive and limited for $\mathcal{L}^n_{\mathrm{BL}}$, and is such that $M \nvDash_{\mathrm{P}} \alpha$. From M, define $M^\tau = \langle W^\tau, \preceq^\tau, v^\tau\rangle$ as described in Step 1. M^τ is defined for $\mathcal{L}^n_{\mathrm{DL}^a}$ and is replete for $\mathcal{L}^n_{\mathrm{BL}}$, Lemma 59. By Lemma 65, $M^\tau \nvDash_{\mathrm{P}} \alpha$. Moreover, $\preceq^\tau$ is reflexive and transitive, Lemma 61, as well as limited for $\mathcal{L}^n_{\mathrm{BL}}$, Lemma 66. From M^τ define R_{M^τ} as described in this Step 2. By Lemma 73, $\mathsf{R}_{M^\tau} \nvDash_{\mathrm{H}} \alpha$. Further, by Lemma 71, R_{M^τ} is reflexive and transitive, and by Lemma 81, R_{M^τ} is limited. Hence, R_{M^τ} is apt for DSDL2. Consequently, $\nVdash_{\mathrm{H}} \alpha$ for this class of relations, a contradiction. Hence, if $\Vdash_{\mathrm{H}} \alpha$ for this class, then $\vdash \alpha$ in DDL-b. Given that R_{M^τ} is transitive, the same can be said with respect to the class of relations that are not only reflexive and limited, but also transitive. ■

Furthermore, for those particularly interested in transitive relations R, since the arguments of Step 1 and Step 2 apply equally well to DDL-a, without the need for the complexity of Lemma 81,

Corollary 83 *DDL-a = DSDL1 is sound and complete for the class of all relations R that are reflexive and transitive.*

We have already seen in Theorem 57 that DDL-c = DSDL2.5 is sound and complete for the class of relations R that are reflexive and transitive as well as stoppered. As a result, there is no principle, or set of principles, in the present framework that demarcates transitivity of relations R, or transitivity with reflexivity, with or without limitation or stoppering. Of course, the combination of transitivity with totality does yield (Rat-Mono) of DDL-d = DSDL3.

6 Ancillary results

With the completion of Step 2 in the argument for DDL-b, and so of all of Stage 2, we have accomplished our primary purpose, proving that DDL-a = DSDL1 (Theorem 52), DDL-b = DSDL2 (Theorem 82), DDL-c = DSDL2.5 (Theorem 57), and DDL-d = DSDL3 (Theorem 10 and its Corollary 12, not to mention Theorem 58). This section presents some further results that follow from those, or the methods that proved them. In particular, in §6.1 we find that DSDL1, DSDL2 and DSDL2.5 are not compact, from which it follows that DDL-a, DDL-b and DDL-c are not strongly complete, and indeed there are no strongly complete axiomatizations for DSDL1, -2 and -2.5. In §6.2 we establish that DDL-a, DDL-b, DDL-c, and DDL-d, are decidable; hence so too are DSDL1, DSDL2, DSDL2.5 and DSDL3. Finally, to end on an optimal note, in §6.3 we examine a variation on the rule Hansson used to interpret formulas $O(B/A)$, a variation that is frequently applied for dyadic deontic logic. This alternative relies on a notion of optimality rather than maximality, such as we have assumed throughout the preceding discussion. Applying our completeness results, however, we discover that, other things being equal, the difference of the interpretive rules makes no difference to the logics themselves.

6.1 Compactness

In Section 4 we proved DDL-d to be strongly complete for the class of DSDL3 models, and hence that DSDL3 is compact, Theorem 10 and its Corollary 13. By contrast, Theorems 52, 82, and 57 establish only weak completeness for DDL-a, DDL-b and DDL-c for their respective classes of H-models. While it may be disappointing not to have the stronger result for the weaker systems, that is too much to ask for. These systems are not strongly complete for those models.

That they are not strongly complete follows from the fact that DSDL1, DSDL2 and DSDL2.5 are not compact. That is to say, there are sets of formulas $\Gamma \subseteq \mathcal{L}_{\mathrm{DL}^a}$ such that every finite subset of Γ is satisfiable by an appropriate model for the system, but Γ itself is not so satisfiable. Here is an example:[9] Let $p, q, r_1, \ldots, r_i, \ldots$ be an enumeration of all the atoms of $\mathcal{L}_{\mathrm{BL}}$. Let

- $\Gamma_0 = \{O(q/p \vee q), P(\neg p \vee \neg q/p), P(p \vee \neg q/\neg(p \leftrightarrow q))\}$,
- $\Gamma_p = \{O(r_i/p) : r_i \neq p \text{ and } r_i \neq q\}$,
- $\Gamma_{\neg(p\leftrightarrow q)} = \{O(r_i/\neg(p \leftrightarrow q)) : r_i \neq p \text{ and } r_i \neq q\}$,
- $\Gamma^\star = \Gamma_0 \cup \Gamma_p \cup \Gamma_{\neg(p\leftrightarrow q)}$.

[9] This is similar to an example Jörg Hansen used to show that his system $DDL^{F,S}$ is not compact with respect to his semantics; cf. [4] p. 496.

Lemma 84 *There is no relation* $\mathsf{R} \subseteq V \times V$ *that satisfies all of* $\Gamma^\star$.

Proof. Suppose, for *reductio*, there were a relation R that satisfied every member of $\Gamma^\star$. Since that includes $P(\neg p \vee \neg q/p)$, there must be some $\varphi \in \mathrm{Max}_{\mathsf{R}}(|p|)$ such that $\varphi \in |\neg p \vee \neg q|$. Since $\varphi \in |p|$, $\varphi \in |\neg q|$, so that $\varphi \notin |q|$. Furthermore, since $O(r_i/p) \in \Gamma^\star$, for every atom r_i other than p or q, $\mathsf{R} \vDash O(r_i/p)$; hence, $\mathrm{Max}_{\mathsf{R}}(|p|) \subseteq |r_i|$, and so $\varphi \in |r_i|$ for all such atoms r_i. Likewise, since $P(p \vee \neg q/\neg(p \leftrightarrow q)) \in \Gamma^\star$, there must be some $\varphi' \in \mathrm{Max}_{\mathsf{R}}(|\neg(p \leftrightarrow q)|)$ such that $\varphi' \in |p \vee \neg q|$. Since $\varphi' \in |\neg(p \leftrightarrow q)|$, $\varphi' \in |\neg p \wedge q|$ or $\varphi' \in |p \wedge \neg q|$. The first is ruled out since $\varphi' \in |p \vee \neg q|$. Hence, $\varphi' \in |p|$ and $\varphi' \notin |q|$. Also, since $O(r_i/\neg(p \leftrightarrow q)) \in \Gamma^\star$, for every atom r_i other than p or q, $\mathsf{R} \vDash O(r_i/\neg(p \leftrightarrow q))$, and so $\mathrm{Max}_{\mathsf{R}}(|\neg(p \leftrightarrow q)|) \subseteq |r_i|$. Hence, $\varphi' \in |r_i|$ for all such atoms, r_i. Thus we see φ and φ' agree on all atoms in $\mathcal{L}_{\mathrm{BL}}$, which means that $\varphi = \varphi'$. Further, although $\varphi \in |p \vee q|$, $\varphi \notin \mathrm{Max}_{\mathsf{R}}(|p \vee q|)$. For, since $O(q/p \vee q) \in \Gamma^\star$, $\mathsf{R} \vDash_{\mathrm{H}} O(q/p \vee q)$, so that $\mathrm{Max}_{\mathsf{R}}(|p \vee q|) \subseteq |q|$. Hence, if $\varphi \in \mathrm{Max}_{\mathsf{R}}(|p \vee q|)$, then $\varphi \in |q|$, whereas already $\varphi \notin |q|$. Since thus $\varphi \notin \mathrm{Max}_{\mathsf{R}}(|p \vee q|)$, there must be a $\varphi'' \in |p \vee q|$ such that $\varphi'' \mathsf{P} \varphi$. $\varphi'' \notin |p|$, for otherwise $\varphi \notin \mathrm{Max}_{\mathsf{R}}(|p|)$. Hence, $\varphi'' \in |q|$, and $\varphi'' \in |\neg p \wedge q|$. In that case, $\varphi'' \in |\neg(p \leftrightarrow q)|$. Since $\varphi = \varphi'$, $\varphi'' \mathsf{P} \varphi'$. But then $\varphi' \notin \mathrm{Max}_{\mathsf{R}}(|\neg(p \leftrightarrow q)|)$, a contradiction. Hence there is no such φ''. But there must be; we are left with a contradiction. Therefore, there is no such relation R that satisfies all of $\Gamma^\star$. ■

Notice this requires no reference to the supposed R being limited or stoppered, or having any other typical traits. It applies to all of the DSDL systems.

Lemma 85 *Every finite subset of* $\Gamma^\star$ *is satisfiable by a relation* $\mathsf{R} \subseteq V \times V$ *that is reflexive or limited or stoppered.*

Proof. Consider an arbitrary finite subset $\Gamma^\star_f$ of $\Gamma^\star$. Given the enumeration $r_1, \ldots, r_i, \ldots$ of all atoms other than p and q, suppose n to be the greatest index occurring on such an atom occurring in a formula in $\Gamma^\star_f$, so that r_{n+1} does not occur in any such formula. Let $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ be those members of V such that

- $\varphi_1(p) = \varphi_1(q) = 1$, and $\varphi_1(r_i) = 1$, for all atoms r_i other than p or q,
- $\varphi_2(p) = 0$ and $\varphi_2(q) = 1$, and $\varphi_2(r_i) = 1$, for all atoms r_i other than p or q,
- $\varphi_3(p) = 1$ and $\varphi_3(q) = 0$, and $\varphi_3(r_i) = 1$, for all atoms r_i other than p or q, including r_{n+1},
- $\varphi_4(p) = 1$ and $\varphi_4(q) = 0$, and $\varphi_4(r_i) = 1$, for all atoms r_i other than p or q, not including r_{n+1}; for it, $\varphi_4(r_{n+1}) = 0$.

Thus, φ_3 and φ_4 agree on all atoms other than r_{n+1}; there they differ. Nonetheless, for all A that might be a component of a member of $\Gamma^\star_f$, $\varphi_3 \in |A|$ iff $\varphi_4 \in |A|$.

Consider the relation $\mathsf{R} \subseteq V \times V$ given entirely by:

- $\varphi_1 \mathsf{R} \varphi_3$,
- $\varphi_2 \mathsf{R} \varphi_4$, and
- for all other $\varphi \in V$, $\varphi_3 \mathsf{R} \varphi$ and $\varphi_4 \mathsf{R} \varphi$,

- for all $\varphi \in V$, $\varphi \mathsf{R} \varphi$.

It is not hard to see that R is reflexive, but more importantly it is stoppered, hence limited. By inspection it is apparent that

- $\mathrm{Max}_{\mathsf{R}}(|p|) = \{\varphi_1, \varphi_4\}$,
- $\mathrm{Max}_{\mathsf{R}}(|\neg(p \leftrightarrow q)|) = \{\varphi_2, \varphi_3\}$,
- $\mathrm{Max}_{\mathsf{R}}(|p \vee q|) = \{\varphi_1, \varphi_2\}$.

That is sufficient to verify that for all $\alpha \in \Gamma^{\star}_{f}$, $\mathsf{R} \models_{\mathrm{H}} \alpha$. Thus $\mathsf{R} \models_{\mathrm{H}} P(\neg p \vee \neg q/p)$, by virtue of φ_4, also $\mathsf{R} \models_{\mathrm{H}} P(p \vee \neg q/\neg(p \leftrightarrow q))$, by virtue of φ_3, and $\mathsf{R} \models_{\mathrm{H}} O(q/p \vee q)$ since both $\varphi_1 \in |q|$ and $\varphi_2 \in |q|$. Hence, if $\alpha \in \Gamma_0$, $\mathsf{R} \models_{\mathrm{H}} \alpha$. If $\alpha \in \Gamma_p$, so that $\alpha = O(r_i/p)$ for some r_i other than p or q, then since $\varphi_1 \in |r_i|$ for all atoms r_i and $\varphi_4 \in |r_i|$ for all atoms r_i except r_{n+1}, and r_{n+1} could not occur in α, then $\mathsf{R} \models_{\mathrm{H}} \alpha$. Similarly in case $\alpha \in \Gamma_{\neg(p \leftrightarrow q)}$ with φ_2 and φ_3. Hence, R satisfies every member of $\Gamma^{\star}_{f}$. ■

From these two lemmas it follows that

Theorem 86 *DSDL1, DSDL2, and DSDL2.5 are not compact with respect to appropriate classes of H-models/relations.*

From that it follows that,

Corollary 87 *DDL-a, DDL-b and DDL-c are not strongly complete with respect to the classes of H-models/relations appropriate for each system. Indeed, no axiomatization of DSDL1, DSDL2, DSDL2.5 is both sound and strongly complete with respect to the classes of H-models/relations appropriate for each system.*

Proof. By the argument for Corollary 13, if there were an axiomatic system **L**, not necessarily DDL-a, -b or -c, that is sound and strongly complete for these models, then DSDL1, -2, or -2.5 would be compact. Since they're not, there is no such **L**. ■

That being so, we will not regret that we have only established the weak completeness of these three systems for their models.

In passing we might note that the relation R defined for Lemma 85 is not total. That is how DSDL3 escapes Theorem 86 and its corollary. As we have seen, DSDL3 is compact, and DDL-d is strongly complete.

The failure of compactness and strong completeness for DSDL1, DSDL2 and DSDL2.5 and their axiomatic equivalents is due to the irredundancy inherent in the framework of H-models; the same would hold for analogous classes of irredundant and replete P-models. By contrast, DDL-a, DDL-b and DDL-c are strongly complete for the appropriate classes of P-models when W may include duplicate members. That can be shown with arguments very like those of Stage 1 in §5.1. Given strong completeness, compactness follows, when redundancy is allowed. Not surprisingly then, there are redundant P-models that satisfy $\Gamma^{\star}$. I leave it as an exercise to find such a model.

6.2 Decidability

Here we show that DDL-a, DDL-b, DDL-c and DDL-d, and their semantical equivalents DSDL1, DSDL2, DSDL2.5 and DSDL3, are decidable. The method to be used is somewhat novel. Other, more familiar procedures, like taking filters through appropriate models, or other similar strategies, that establish systems to have the finite model property, from which decidability would follow, given the systems' finite axiomatizability, do not apply to these logics. At least, DDL-b, DDL-c and DDL-d do not have the finite model property.

Indeed, these systems have what might be called the 'infinite model property'. Due to (RP), they are sound for no class of models that contains even one finite member. I leave it as an exercise to verify that there is no finite P-model that satisfies all the theorems of DDL-b, DDL-c or DDL-d, where I refer to P-models simply because H-models must be understood as relations R over the infinite set of valuations V. If one allows relations over finite subsets of V, then the same would obtain; there is no relation over a finite subset of V that satisfies all of DDL-b, DDL-c or DDL-d.

Given the failure of the finite model property for these systems, we now draw their decidability instead from the fact that they are conservative extensions of their finite counterparts DDL^n-a, -b, -c, -d, which, as we have seen, do possess the finite model property and are decidable.

In general, a logical system $\mathbf{L}_1$ is said to be a conservative extension of another, $\mathbf{L}_2$, just in case $\mathbf{L}_2 \subseteq \mathbf{L}_1$ (extension) and also for every $\phi \in \mathbf{L}_1$ that is in the vocabulary of $\mathbf{L}_2$, $\phi \in \mathbf{L}_2$ (conservation). Here we find that, for any finite $n \geq 1$, the full axiomatically defined DDL systems are conservative extensions of their finite DDL^n counterparts.

If $\mathbf{L}$ is any of the systems DDL-a, DDL-b, DDL-c, or DDL-d, and $\mathbf{L}^n$ is its finite counterpart in $\mathcal{L}^n_{\mathrm{DL}^a}$, for finite $n \geq 1$:

Theorem 88 *$\boldsymbol{L}$ is a conservative extension of $\boldsymbol{L}^n$.*

Proof. Lemma 14 provides that $\mathbf{L}$ is an extension of $\mathbf{L}^n$. For conservation, suppose $\alpha \in \mathcal{L}^n_{\mathrm{DL}^a}$, and suppose $\vdash \alpha$ for any of these full systems, $\mathbf{L}$. Suppose, however, for *reductio*, $\nvdash \alpha$ for $\mathbf{L}^n$, the finite counterpart of $\mathbf{L}$ in $\mathcal{L}^n_{\mathrm{DL}^a}$. By the completeness results of Stage 1, §5.1, there is a P-model, M, of the appropriate kind such that $M \not\models_{\mathrm{P}} \alpha$. From such an M, through the procedures of Stage 2, §5.2, we find a relation $\mathsf{R} = \mathsf{R}_M$, or $\mathsf{R} = \mathsf{R}_{M^r}$ in the case of DDL-b, of the kind appropriate for $\mathbf{L}$ such that $\mathsf{R} \not\models_{\mathrm{H}} \alpha$. Hence, $\not\Vdash_{\mathrm{H}} \alpha$ for that class of models. By the soundness of $\mathbf{L}$, however, $\Vdash_{\mathrm{H}} \alpha$, a contradiction. Hence, $\vdash \alpha$ for $\mathbf{L}^n$. ■

As an immediate consequence, we have

Corollary 89 $\boldsymbol{L}^n = \boldsymbol{L} \cap \mathcal{L}^n_{\mathrm{DL}^a}$.

which was briefly mentioned early in §5.

Related to this, we can see that each of the full systems $\mathbf{L}$ is nothing but the sum of its finite subsystems. That is,

Corollary 90 $\boldsymbol{L} = \bigcup_{n=1}^{\infty} \boldsymbol{L}^n$

Proof. That $\bigcup_{n=1}^{\infty} \mathbf{L}^n \subseteq \mathbf{L}$, follows from Lemma 14. For the converse, consider any α such that $\vdash \alpha$ in $\mathbf{L}$. Since there is a finite $n \geq 1$ such that $\lambda(\alpha) = n$, so that $\alpha \in \mathcal{L}^n_{\mathrm{DL}^a}$, then $\vdash \alpha$ in $\mathbf{L}^n$, since $\mathbf{L}$ is a conservative extension of $\mathbf{L}^n$. Hence, $\alpha \in \bigcup_{n=1}^{\infty} \mathbf{L}^n$, which suffices for $\mathbf{L} \subseteq \bigcup_{n=1}^{\infty} \mathbf{L}^n$. ■

Importantly, Theorem 88 also provides for the decidability of these systems.[10]

Theorem 91 *(i) DDL-a, DDL-b, DDL-c, and DDL-d are decidable. (ii) DSDL1, DSDL2, DSDL2.5 and DSDL3 are decidable.*

Proof. For (i), let $\mathbf{L}$ be any of DDL-a, -b, -c or -d, and consider an arbitrary $\alpha \in \mathcal{L}_{\mathrm{DL}^a}$. There is some finite $n \geq 1$ such that $\lambda(\alpha) = n$, and that is decidable. Whether $\vdash \alpha$ in $\mathbf{L}^n$ or $\nvdash \alpha$ in $\mathbf{L}^n$, is decidable, Corollaries 30, 41, and 46. If $\vdash \alpha$ in $\mathbf{L}^n$, then $\vdash \alpha$ in $\mathbf{L}$, Lemma 14, and if $\nvdash \alpha$ in $\mathbf{L}^n$ then $\nvdash \alpha$ in $\mathbf{L}$, since $\mathbf{L}$ is a conservative extension of $\mathbf{L}^n$, Theorem 88. Hence the decision for the finite subsystem extends to the full system. For (ii), apply the equivalences of the systems. ■

6.3 Maximality and optimality

In Footnote 3, I observed that in his treatment of DSDL3, Spohn, [16], p. 239, applied a definition of maximality different from Hansson's own. Others too use this other definition, e.g., Åqvist [17, 18], and Parent [10, 12], although they apply it more in the framework of P-models than Hanssonian H-models. I present that other definition here, and show that, even though the concepts are not equivalent, the difference makes no difference for the sets of valid formulas generated, other things being equal. To ease discussion I now reserve the term 'maximality' for Hansson's original stipulation and call the other concept 'optimality'.[11]

For any relation $\mathsf{R} \subseteq V \times V$, and any $X \subseteq V$, let us say φ is optimal in X by R, or R-optimal, just in case φ is ranked at least as highly as any other $\varphi' \in X$, i.e.,

- $\mathrm{Opt}_{\mathsf{R}}(X) = \{\varphi : \varphi \in X \text{ and for all } \varphi' \in X, \varphi \mathsf{R} \varphi'\}$,

which we may contrast with the original:

- $\mathrm{Max}_{\mathsf{R}}(X) = \{\varphi : \varphi \in X \text{ and there is no } \varphi' \in X \text{ such that } \varphi' \mathsf{P} \varphi\}$, or equivalently,

  $\mathrm{Max}_{\mathsf{R}}(X) = \{\varphi : \varphi \in X \text{ and for all } \varphi' \in X, \text{ if } \varphi' \mathsf{R} \varphi, \text{ then } \varphi \mathsf{R} \varphi'\}$.

In general, for any $X \subseteq V$, $\mathrm{Opt}_{\mathsf{R}}(X) \subseteq \mathrm{Max}_{\mathsf{R}}(X)$, but not conversely. If, however, R is total, then $\mathrm{Opt}_{\mathsf{R}}(X) = \mathrm{Max}_{\mathsf{R}}(X)$.

With this alternative idea of what it means to be 'best', one can interpret formulas $O(B/A)$ by an optimality rule:

- Rule O $\quad \mathsf{R} \models_{\mathrm{O}} O(B/A)$ iff $\mathrm{Opt}_{\mathsf{R}}(|A|) \subseteq |B|$.

[10] Spohn, [16], p. 251, also demonstrated that DSDL3 is decidable, applying a different method.

[11] Parent [11, 12, 13] also discusses the contrast between maximality and optimality in a similar vein.

We define O-validity as with H-validity but with Rule O applied in place of Rule H. Thus, $\alpha \in \mathcal{L}_{\mathrm{DL}^a}$ is O-valid with respect to a class of relations R just in case, for every R in that class $\mathsf{R} \models_{\mathrm{O}} \alpha$, and so on for related concepts.

Likewise one can specify when relations are limited or stoppered in terms of this notion.

- R is O-limited iff, for all $A \in \mathcal{L}_{\mathrm{BL}}$, if $|A| \neq \varnothing$, then $\mathrm{Opt}_{\mathsf{R}}(|A|) \neq \varnothing$.
- R is O-stoppered iff, for all $A \in \mathcal{L}_{\mathrm{BL}}$, if $\varphi \in |A|$ then either $\varphi \in \mathrm{Opt}_{\mathsf{R}}(|A|)$ or there is a $\varphi' \in \mathrm{Opt}_{\mathsf{R}}(|A|)$ such that $\varphi' \mathsf{P} \varphi$.

On the basis of those, we define variations on Hansson's DSDL systems.

- $\mathrm{DSDL}^{\mathrm{o}}1$ is the set of $\alpha \in \mathcal{L}_{\mathrm{DL}^a}$, that are O-valid with respect to the class of all reflexive relations $\mathsf{R} \subseteq V \times V$;
- $\mathrm{DSDL}^{\mathrm{o}}2$ is the set of $\alpha \in \mathcal{L}_{\mathrm{DL}^a}$, that are O-valid with respect to the class of all relations $\mathsf{R} \subseteq V \times V$ that are reflexive and O-limited;
- $\mathrm{DSDL}^{\mathrm{o}}2.5$ is the set of $\alpha \in \mathcal{L}_{\mathrm{DL}^a}$, that are O-valid with respect to the class of all relations $\mathsf{R} \subseteq V \times V$ that are reflexive and O-stoppered;
- $\mathrm{DSDL}^{\mathrm{o}}3$ is the set of $\alpha \in \mathcal{L}_{\mathrm{DL}^a}$, that are O-valid with respect to the class of all relations $\mathsf{R} \subseteq V \times V$ that are O-limited and also transitive and total.

Given the difference between optimality and maximality, one might expect the $\mathrm{DSDL}^{\mathrm{o}}$ systems to differ from Hansson's own DSDL systems. That is not the case, however.

Although, in general, $\mathrm{Opt}_{\mathsf{R}}(X) \neq \mathrm{Max}_{\mathsf{R}}(X)$, when R is not total, nevertheless, for any R, there is another relation, R^{o}, close by such that $\mathrm{Max}_{\mathsf{R}}(X) = \mathrm{Opt}_{\mathsf{R}^{\mathrm{o}}}(X)$. Given R, define a relation, I, of incomparability by R, and then define R^{o}, thus:

- $\varphi \mathsf{I} \varphi'$ iff not-($\varphi \mathsf{R} \varphi'$) and not-($\varphi' \mathsf{R} \varphi$);
- $\varphi \mathsf{R}^{\mathrm{o}} \varphi'$ iff either $\varphi \mathsf{R} \varphi'$ or $\varphi \mathsf{I} \varphi'$.

It is easy to see that R^{o} is total. More to the point,

Lemma 92 *For any $X \subseteq V$,* $\mathrm{Max}_{\mathsf{R}}(X) = \mathrm{Opt}_{\mathsf{R}^{\mathrm{o}}}(X)$.

Proof. Suppose $\varphi \in \mathrm{Max}_{\mathsf{R}}(X)$, so that $\varphi \in X$ and there is no $\varphi' \in X$ such that $\varphi' \mathsf{P} \varphi$. To show that $\varphi \in \mathrm{Opt}_{\mathsf{R}^{\mathrm{o}}}(X)$, we need to show that for all $\varphi' \in X$, $\varphi \mathsf{R}^{\mathrm{o}} \varphi'$. To that end, suppose $\varphi' \in X$, but, for *reductio*, not-($\varphi \mathsf{R}^{\mathrm{o}} \varphi'$). Then it is not the case that either $\varphi \mathsf{R} \varphi'$ or $\varphi \mathsf{I} \varphi'$, i.e., not-($\varphi \mathsf{R} \varphi'$) and also not-($\varphi \mathsf{I} \varphi'$). By the latter, $\varphi \mathsf{R} \varphi'$ or $\varphi' \mathsf{R} \varphi$. Since not the first, $\varphi' \mathsf{R} \varphi$. Since thus $\varphi' \mathsf{R} \varphi$ and not-($\varphi \mathsf{R} \varphi'$), $\varphi' \mathsf{P} \varphi$, in which case $\varphi \notin \mathrm{Max}_{\mathsf{R}}(X)$, a contradiction. Hence, $\varphi \mathsf{R}^{\mathrm{o}} \varphi'$, which suffices for $\varphi \in \mathrm{Opt}_{\mathsf{R}^{\mathrm{o}}}(X)$.

For the converse, suppose $\varphi \in \mathrm{Opt}_{\mathsf{R}^{\mathrm{o}}}(X)$, so that $\varphi \in X$ and for all $\varphi' \in X$, $\varphi \mathsf{R}^{\mathrm{o}} \varphi'$. To show $\varphi \in \mathrm{Max}_{\mathsf{R}}(X)$, suppose, for *reductio*, there is a $\varphi' \in X$ such that $\varphi' \mathsf{P} \varphi$. Since $\varphi' \in X$, $\varphi \mathsf{R}^{\mathrm{o}} \varphi'$. Hence, either $\varphi \mathsf{R} \varphi'$, or $\varphi \mathsf{I} \varphi'$. Not the first, since $\varphi' \mathsf{P} \varphi$, but also not the second, since $\varphi' \mathsf{R} \varphi$. Hence, there is no such φ', and so $\varphi \in \mathrm{Max}_{\mathsf{R}}(X)$. ∎

Lemma 93 *(i) If* R *is limited, then* R^o *is O-limited; (ii) if* R *is stoppered, then* R^o *is O-stoppered. (iii) If* R *is both transitive and total, then* R^o *is both transitive and total.*

Proof. (i) and (ii) are immediate from Lemma 92. For (iii), if R is transitive and total, then $\mathsf{R} = \mathsf{R}^o$, so of course R^o is transitive and total. ■

From this we find that R and R^o are equivalent, by their respective modellings, in the sense that:

Lemma 94 *For all* $\alpha \in \mathcal{L}_{\mathrm{DL}^a}$, $\mathsf{R} \models_{\mathrm{H}} \alpha$ *iff* $\mathsf{R}^o \models_{\mathrm{O}} \alpha$.

Proof. Proof is by induction on α. We show only the basis case, where $\alpha = O(B/A)$, for some $A, B \in \mathcal{L}_{\mathrm{BL}}$, since the induction to more complex α is routine and easy. Given Lemma 92, $\mathsf{R} \models_{\mathrm{H}} O(B/A)$ iff $\mathrm{Max}_{\mathsf{R}}(|A|) \subseteq |B|$ iff $\mathrm{Opt}_{\mathsf{R}^o}(|A|) \subseteq |B|$ iff $\mathsf{R}^o \models_{\mathrm{O}} O(B/A)$. ■

Let **L** be any of the systems DDL-a, DDL-b, DDL-c or DDL-d, then **L** is sound with respect to the classes of relations apt for **L** when validity is understood as O-validity, and formulas are interpreted by way of Rule O. I.e.,

Lemma 95 *For all* $\alpha \in \mathcal{L}_{\mathrm{DL}^a}$*, if* $\vdash \alpha$ *in* ***L****, then* α *is O-valid with respect to the classes of models,* R*, apt for* ***L****, i.e., those that are O-limited for DDL-b, O-stoppered for DDL-c, and O-limited, transitive and total for DDL-d.*

Proof. Easily shown, in the usual way, and so left to the reader. ■

With our earlier completeness results, it is now not difficult to establish that the DSDL^o systems, defined in terms of Rule O, are equivalent to the original DSDL systems, defined in terms of Rule H.

Theorem 96 *(i) DSDL^{o}1 = DSDL1; (ii) DSDL^{o}2 = DSDL2;*
(iii) DSDL^{o}2.5 = DSDL2.5; (iv) DSDL^{o}3 = DSDL3.

Proof. For (i) suppose $\alpha \in$ DSDL^{o}1, but, for *reductio*, $\alpha \notin$ DSDL1. Then there is a reflexive R such that $\mathsf{R} \not\models_{\mathrm{H}} \alpha$. From R define R^o, as described. By Lemma 94, $\mathsf{R}^o \not\models_{\mathrm{O}} \alpha$, in which case $\alpha \notin$ DSDL^{o}1, a contradiction. Hence, $\alpha \in$ DSDL1, and thus DSDL^{o}1 $\subseteq$ DSDL1. For the converse, suppose $\alpha \in$ DSDL1. Then $\vdash \alpha$ in DDL-a, by the completeness of that system, Theorem 52. Hence, α is O-valid for the class of all relations R, by the O-soundness of DDL-a, Lemma 95. I.e., $\alpha \in$ DSDL^{o}1, which suffices for DSDL1 $\subseteq$ DSDL^{o}1, and thus DSDL^o = DSDL1. For (ii), (iii) and (iv) the argument is similar, applying Lemma 93 as required. ■

With that, we find that, as far as what formulas are valid under the rules, the choice between Hansson's original Rule H and the alternative Rule O makes no difference, other things being equal. It might, of course, make a difference with regard to what one says about particular models, or the role of various properties of those models. For example, Spohn [16] §4.2 can argue that the condition of totality is idle for the determination of DSDL3, but only because he is applying Rule O, rather than Hansson's Rule H. Under Rule H, it is necessary to require relations R to be total in order to validate (RatMono) or Spohn's own axiom (A4).

7 Quick recap

Hansson defined his dyadic deontic logics, DSDL1, DSDL2 and DSDL3, entirely semantically, as sets of formulas valid with respect to certain classes of models, construed as relations R over the set V of classical valuations. To his three, I have added another, DSDL2.5, between DSDL2 and DSDL3; for all of four, see §2. The purpose of this paper was to provide an axiomatization for each of them. For that, in §3, I introduced the axiomatic systems, DDL-a, DDL-b, DDL-c and DDL-d. In §4 and §5 these were proved to be sound and complete with respect to the classes of models for DSDL1, DSDL2, DSDL2.5 and DSDL3 respectively, and so to be equivalent to Hansson's semantical systems. That accomplished our primary goal. For a lagniappe, we found at the end of §5 that under Rule H relations R can be required to be transitive without affecting what principles are valid.

In §4 we proved DDL-d to be strongly complete for DSDL3 models, and hence that DSDL3 is compact. By contrast, in §5 we only proved the three weaker systems to be weakly complete. In §6.1 we found that DSDL1, -2, and -2.5 are not compact, and so there is no strongly complete axiomatization for them. In addition, in §6.2 we established that each of the axiomatic systems and its semantical counterpart is decidable, by virtue of the DDL systems being conservative extensions of their decidable finite subsystems, DDL^n, for $n \geq 1$. Finally, in §6.3, we saw that, somewhat surprisingly, as regards which formulas are valid, other things being equal, it does not matter whether one interprets formulas $O(B/A)$ in terms of maximality with Rule H, as Hansson did, or in terms of optimality with Rule O, as others have.

To demonstrate these results proved more challenging than one might have expected at first, at least for the three weaker logics. Inevitably there are complexities in treating these systems, as we saw in §5.1 with Stage 1 of the argument. The challenge is exacerbated, however, by two particular features of Hansson's systems and their models. One must accommodate the infinitude of the languages $\mathcal{L}_{\mathrm{BL}}$ and $\mathcal{L}_{\mathrm{DL}^a}$, and one must provide models for these languages that are irredundant, as the class of classical valuations, V, over which Hansson's relations R are defined, necessarily is. If it were not for the call for irredundancy we could have stopped at Stage 1, as it is easy to apply the demonstrations there to the full axiomatic systems in the infinite language.

To obtain the requisite irredundant models, it was especially helpful to begin by retreating from the infinite language, to establish results first for finite counterparts of DDL-a, -b and -c in terms of finite P-models. This was crucial for the decidability results mentioned above. More to the point, this allowed us in Stage 2 of §5.2 to apply the method of marking worlds from the original P-models, and so to derive the irredundant H-models required for the demonstrations of completeness for the full infinite DDL logics, which proved their equivalences to Hansson's DSDL systems.

If we can achieve completeness results both for the finite DDL^n logics and for the full infinite DDL systems with respect to redundant models, and if we can achieve completeness results for the full infinite systems with respect to irredundant models, what then of irredundant models for the finite logics? In the framework of Hansson's semantics, consider the class, V^n, of valuations defined only for the n many atoms of the finite language $\mathcal{L}^n_{\mathrm{BL}}$, and relations $\mathsf{R}^n \subseteq V^n \times V^n$ defined over them. V^n is necessarily irredundant. One might then expect DDL^n-a to be equivalent to DSDL^{n}1, the formulas valid for all such relations R^n, and DDL^n-b to be equivalent to DSDL^{n}2, those valid for all such R^n that are limited for $\mathcal{L}^n_{\mathrm{BL}}$, and DDL^n-c equivalent to DSDL^{n}2.5, those valid for all R^n that are stoppered for $\mathcal{L}^n_{\mathrm{BL}}$. One might expect that, but it is not so, at least not when $n \geq 2$, and it is not so precisely because of the irredundancy inherent in these models, though interestingly DDL^n-d is equivalent to DSDL^{n}3, those formulas valid for $\mathsf{R}^n \subseteq V^n \times V^n$ that are limited for $\mathcal{L}^n_{\mathrm{BL}}$, transitive and total. I will not explore this topic here, however. Instead, I leave it as an open, educational exercise for the intrepid reader to find how those equivalences fail, and then to devise complete axiomatizations for finite DSDL^{n}1, DSDL^{n}2 and DSDL^{n}2.5. It can be done.

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